# DISTINGUISHED REPRESENTATIONS FOR $GL(n)$

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*To Acchan and Amma*

*with love and admiration*

# Certificate

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This is to certify that Mr. U. K. **Anandavardhanan** has carried out the research work embodied in the present thesis titled **DISTINGUISHED REPRESENTATIONS FOR GL(n)** for the full period prescribed under the University rules. No part of this thesis was earlier submitted for the award of research degree of any University/Institute.

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# CONTENTS

# 1 INTRODUCTION

## 1.1 Notation

Let $F$ be a *non-discrete locally compact totally disconnected topological field* (a *non-Archimedean local field*) of characteristic zero. Thus $F$ is a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_F$ be the ring of integers of $F$ and $\mathcal{P}_F$ the unique maximal ideal of $\mathcal{O}_F$. The group of units of $\mathcal{O}_F$, which is $\mathcal{O}_F - \mathcal{P}_F$, is denoted by $\mathcal{O}_F^\times$. Let $\varpi_F$ be a generator of $\mathcal{P}_F$. It is called a *uniformizer* of $F$. Let $v_F$ be the *valuation* of $F$ such that $v_F(\varpi_F) = 1$. The cardinality of $\mathcal{O}_F/\mathcal{P}_F$ is denoted by $q_F$, where $q_F$ is a power of $p$. Denote by $|\ |$ the (normalized) *absolute value* of $F$ such that $|a| = q_F^{-v_F(a)}$ for $a \in F$. We fix a nontrivial additive character $\psi$ of $F$. The *conductor* of $\psi$ is the largest (fractional) ideal $\mathcal{P}_F^{n(\psi)}$ of $\mathcal{O}_F$ on which $\psi$ is trivial. Then $n(\psi)$ is called the *conductor exponent* of $\psi$. For $a \in F$, let $\psi_a$ stand for the additive character given by

$$\psi_a(x) = \psi(ax)$$

Then $n(\psi_a) = n(\psi) + v_F(a)$. The *Schwartz-Bruhat space* of $F$ consists of locally constant compactly supported functions on $F$ and is denoted by $\mathcal{S}(F)$. If $\Phi$ belongs to $\mathcal{S}(F)$, then the *Fourier transform* of $\Phi$ with respect to $\psi$ is given by

$$\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)\,dy$$

where $dy$ is an additive Haar measure on $F$. The measure $dy$ is said to be *self-dual* (with respect to $\psi$) if $\Phi(x) = \Phi(-x)$. The volume of $\mathcal{O}$ with respect to the self-dual measure turns out to be $q_F^{n/2}$.

Let $F$ denote the multiplicative group of $F$. Then

$$\mathcal{O}_F \supset 1+\mathcal{P}_F \supset 1+\mathcal{P}_F \supset \quad \supset 1+\mathcal{P}_F \supset$$

form a *fundamental system of open neighbourhoods at identity* for $F$. If $\mu$ is a character of $F$, then the *conductor* of $\mu$ is the largest $1+\mathcal{P}_F$ on which $\mu$ is trivial. This $n$ is called the *conductoral exponent* of $\mu$. We denote it by $a(\mu)$.

Let $GL_n(F)$ denote the *general linear group* of $n \times n$ matrices of non-zero determinant. We consider *irreducible admissible representations* of $GL_n(F)$. For such a representation $\pi$, $\pi$ denotes the representation *contragredient* to $\pi$ and we let $\omega$ denote the *central character* of $\pi$. A character $\mu$ of $F$ gives rise to a one-dimensional representation $\mu \circ det$ of $GL_n(F)$, where $det$ stands for the *determinant map*. We shall denote this one-dimensional representation by $\mu$ again. There is a *Gauss sum* attached to an irreducible admissible representation $\pi$ and the character $\psi$, which is denoted by $\epsilon(s, \pi, \psi)$, where $s$ is a complex variable. More generally we have such an *epsilon factor* for a pair $(\pi_1, \pi_2)$ of representations of $GL_n(F)$, $i = 1, 2$, denoted $\epsilon(s, \pi_1 \times \pi_2, \psi)$. There is a similar $\gamma$ factor which is related to the $\epsilon$ factor. Precise details are given in the next chapter.

Now let $E$ be a quadratic extension of $F$. All the preceding considerations apply over the field $E$ with the obvious changes. Let $\sigma$ denote the non-trivial element of $Gal(E/F)$, the *Galois group* of $E/F$. Denote by $N_{E/F}$ and $Tr_{E/F}$ the *norm map* and the *trace map* from $E$ to $F$. Let [illegible] denote the non-trivial character of $F / N_{E/F}(E)$

by $\omega_E$. The additive character of $E$ denoted by $\psi$ that we choose is such that it has trivial restriction to $F$. Let $\psi_E$ be the character of $E$ defined by

$$\psi_E(x) = \psi(Tr_{E/F}(x))$$

Then $\psi$ is of the form $(\psi_E)_\Delta$ for a character $\psi$ of $F$ and $\Delta \in E$ with $Tr_{E/F}(\Delta) = 0$. A representation of $GL_n(E)$ is typically denoted by $\Pi$. Associated to such a representation $\Pi$ we have a *twisted epsilon factor* which we denote by $\epsilon(s, r(\Pi), \psi)$ and a *twisted L function* $L(s, r(\Pi))$. Other than the general linear groups we also look at (in chapter 4) the *unitary group* in two variables, denoted by $U(2, E/F)$. In general $U(n, E/F)$ is defined as the group of those elements in $GL_n(E)$ that are fixed by an involution $\tau$ which is defined in terms of $\sigma$ (see 2.3).

Finally we introduce the main theme of this dissertation. Let $\mu$ be a character of $F$. A representation $(\Pi, V)$ of $GL_n(F)$ is said to be $\mu$-*distinguished with respect to* $GL_n(F)$ if there is a non zero linear form $l$ on $V$, the space of $\Pi$, such that

$$l(\Pi(g)v) = \mu(\det g)\, l(v)$$

for all $g \in GL_n(F)$ and $v \in V$. When we omit the reference to a subgroup of $GL_n(E)$, it is understood that the subgroup is $GL_n(F)$. The representation is said to be *distinguished* if it is 1-distinguished. Thus if $H$ is a subgroup of $GL_n(E)$, a representation $\Pi$ of $GL_n(F)$ is distinguished with respect to $H$ (or $H$-distinguished) if the space of $H$-invariant linear forms on $\Pi$, denoted by $Hom_H(\Pi, 1)$, is nonzero. In this thesis $H$ will usually be $GL_n(F)$ except in chapter 4 where it is $U(2, E/F)$.

The notion of a *base change lift* is introduced in 2.3.

## 1.2 Motivation and some remarks

Suppose one considers an algebraic group $G$ over a global field and an algebraic subgroup $H$ of $G$. Then distinguishedness of an *automorphic representation* of $G$ is defined in terms of the non-vanishing of a certain *period integral*. This notion was introduced in the work of Harder, Langlands and Rapoport [22] where they prove *Tate's conjectures* for certain *Hilbert modular surfaces*. In order to do this they need to investigate the poles of the *Hasse-Weil zeta function* attached to the surface. The zeta function is expressed in terms of some integrals which have poles with residue equal to the value of the period integral. Thus distinguishedness determines the poles of the zeta function.

If $E/F$ is a quadratic extension of global fields, let $\mathbf{A}$ and $\mathbf{A}_E$ denote the *ring of adeles* of $F$ and $E$ respectively. Then it is proved in [22] that a cuspidal automorphic representation of $GL_2(\mathbf{A}_E)$ with trivial central character is distinguished if and only if it is the base change lift of a cuspidal representation of $GL_2(\mathbf{A}_F)$ whose central character is non-trivial. Alternate proofs of this global result appeared in Flicker [12] and Ye [36]. Flicker uses *twisted tensor L-functions* whereas Ye uses *relative trace formulas* [26] to achieve this. There are several other instances in the literature where a similar phenomenon, that distinguished representations have a functorial characterization, takes place. To cite a few which are relevant to this thesis we refer to [13], [21], [27], [28] and [30].

In this thesis we deal with three such instances, all in the context of $GL(2)$ over a $p$-adic field. These are the equivalence of (1) and (3) in theorem 1.3.1, equivalence of (1) and (2) in theorem 1.3.2 and theorem 1.3.3. The first is the local analogue of the theorem in [22] cited in the above paragraph and in fact our result can be deduced from its global analogue. This was pointed out to us by the referee of [1]. One starts with a local distinguished representation (assume it is supercuspidal which is the non trivial case anyway) and shows that it can be realized as a local component of a global distinguished representation. By the theorem in [22], this global representation is a base change lift. Thus its components are local base change lifts with the desired central characters. The converse is similar using the fact that a local base change lift may be realized as a component of a global base change lift.

The key ingredients in our proof (of theorem 1.3.1) are the $\gamma$ factor criterion for distinguishedness due to Hakim (Theorem 3.1.4), Tunnell's formula for characters of $GL(2)$ (Theorem 3.1.8) and a result of Saito which is a corollary to his proof of Tunnell's formula (Theorem 3.1.9). We reproduce Hakim's theorem and his proof (with minor changes) in 3.1. Then we derive two nice results (corollaries 3.1.6, 3.1 [illegible]) about the 1-dimensional characters of $E$ out of Hakim's criterion fairly easily. One is a well known result due to Frohlich and Queyrut [18] and the other result is stated in [3]. Theorems 1.3.2, 1.3.3 and 1.3.4 are deduced from theorem 1.3.1 without much effort.

Theorem 1.3.5 is due to Flicker. It is the main local theorem in [13] where he deduces this from a similar global result. We give a purely local proof of this result in chapter 4. More details regarding the theorem and our proof are given in sections 4.1 and 4.2.

In the final chapter we prove a multiplicity one result for a class of representations of $GL_n(E)$ and this is stated as theorem 1.3.6. It follows essentially from the [illegible] lemma in [15]. We use this to give a characterization for distinguishedness for these representations (Theorem 1.3.7). We also try to understand how distinguishedness reflects on the local twisted tensor $L$ function (Theorem 1.3.8). Theorem 1.3.9 is easy to verify once we appeal to a result due to Bushnell and Henniart.

The existing proofs of the known theorems in this thesis (addressed in chapters 3 and 4) use a mixture of local and global methods. We provide local proofs in those instances where only global proofs currently exist. We believe this results in understanding the situation better. It appears that the results in the final chapter are new. Nevertheless they are not comprehensive and should be improved upon. We only remark on a few plausible improvements here.

## 1.3 Summary of results

Theorems 1.3.1-1.3.4 are taken up in chapter 3. Chapter 4 is devoted to theorem 1.3.5. The rest are addressed in chapter 5.

Theorem 1.3.1. Let $\mu$ be a character of $F$. Let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$. Then the following statements are equivalent.

(1) $\Pi$ is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{E/F}$.
(2) $\gamma(\Pi \otimes \lambda^{-1}, \psi'_E)\lambda(-1) = 1$ for all characters $\lambda$ of $E$ which satisfy $\lambda|_F = \mu$.
(3) $\Pi$ is $\mu$-distinguished with respect to $GL_2(F)$.

Theorem 1.3.2. Let $\mu$ be a character of $F$ and $\Pi$ an irreducible admissible representation of $GL_2(E)$ with central character $\omega_\Pi = \mu \circ N_{E/F}$. Then the following are equivalent.

(1) $\Pi \sim \Pi^\sigma$ i e $\Pi$ is a base change lift from $GL_2(F)$

(2) $\Pi$ is distinguished with respect to $U(2\ E/F)$

(3) $\Pi$ is $\mu$-distinguished or $\mu\omega_{E\ F}$ distinguished

Theorem 1 3 3 Let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi|_F = 1$ Then $\Pi \sim \Pi^\sigma$ if and only if $\Pi$ is distinguished or $\omega_F$ distinguished with respect to $GL_2(F)$

Theorem 1 3 4 Let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ Then $\Pi$ is both distinguished and $\omega_{E\ F}$ distinguished with respect to $GL_2(F)$ exactly when $\Pi = \Pi(\chi\ \chi\ )$ for some character $\chi$ of $E$

Theorem 1 3 5 An irreducible admissible representation $\Pi$ of $GL_2(E)$ is distinguished with respect to $GL\ (F)$ if and only if it is an unstable base change lift of a representation of $U(2\ E/F)$

Theorem 1 3 6 Let $\Pi$ be a distinguished supercuspidal representation of $GL_n(E)$ or $\Pi\quad I(\Pi\ \ \Pi_2)$ is normalizedly induced from the standard parabolic of type $(n_1\ n_2)$ and $\Pi$ are supercuspidals of $GL_n\ (E)$ distinguished with respect to $GL_n\ (F)$ $1\ 2$ Then

$$dim_{\mathbb{C}}(Hom_{P_n}\ (\pi\ 1)) = 1$$

where $P_n(F)$ is the mirabolic subgroup of $F$

Theorem 1 3 7 Let $\Pi$ be a supercuspidal representation of $GL_n(E)$ or $\Pi\quad I(\Pi\ \ \Pi_2)$ where $\Pi_1$ and $\Pi_2$ are supercuspidals of $GL_n\ (E)$ and $GL\ \ (F)$ respectively with $n = n\ + n_2$ Suppose $\omega_\Pi\quad 1$ Then $\Pi$ is distinguished if and only if

$$\int_{N\ F\ P} W\ p)d\ p = \int_{\ F\ P\ F} W\ (p)d\ p$$

for all $W \in \mathcal{W}(\Pi, \psi)$

Theorem 1.3.8 Let $\Pi$ be as in theorem 1.3.6. Then $s = 0$ is a pole of $L(s, (\Pi))$

Theorem 1.3.9 Let $\Pi$ be a tempered representation of $GL_n(E)$. Then

$$\epsilon(\frac{1}{2}, \Pi \times \Pi^{\sigma}, \psi) = 1$$

Suppose $\Pi$ is tempered and distinguished. Then

$$\epsilon(\frac{1}{2}, \Pi \times \Pi, \psi) = 1$$

# 2 PRELIMINARIES

## 2.1 Representations of $GL_n(F)$

The references for this section are mainly [5], [6], [7], [10] and [31]. We have particularly made use of the excellent expositions in [10] and [31].

Let $F$ be a non-Archimedean local field of characteristic zero. The $\mathcal{O}_F$, $\mathcal{P}_F$, $\varpi_F$, $q_F$ have their usual meanings as in 1.1. Let $M_n(F)$ denote the set of all $n \times n$ matrices whose entries are in $F$ and $GL_n(F)$ the general linear group of invertible matrices in $M_n(F)$. It is a locally compact totally disconnected topological group. Its unique (upto conjugacy) maximal open compact subgroup is $K = GL_n(\mathcal{O}_F)$. For a fundamental system of open neighbourhoods at the identity, we can take the chain of open compact subgroups given by

$$K_m = \{g \in K \mid g - 1_n \in \varpi^m M_n(\mathcal{O}_F)\}$$

for $m \geq 1$ and $K_0 = K$. Let $B_n(F)$, $N_n(F)$, $A_n(F)$ denote the Borel subgroup of upper triangular matrices, its nilpotent radical which consists of upper triangular unipotent matrices and the diagonal torus with entries all zero except on the diagonal. The $A_n$ normalizes $N_n$ and we have $B_n = N_n A_n$. Let $S_n$ be the Weyl group of all [illegible] symmetric [illegible]s. We will need the following decompositions

Theorem 2.1.1 (Bruhat decomposition)

$$GL_n(F) = \bigcup_{w\in S} B_n(F)wB_n(F) = \bigcup_{w\in S} B_n(F)wN_n(F)$$

Corollary 2.1.2 $GL_n(F)$ is generated by $B_n(F)$ and its transpose.

Theorem 2.1.3 (Iwasawa decomposition)

$$GL_n(F) = N_n(F)A_n(F)K$$

Theorem 2.1.4 (Iwahori factorization) For $m \geq 1$,

$$K_m = \left(K_m \bigcap N_n(F)\right)\left(K_m \bigcap A_n(F)\right)\left(K_m \bigcap {}^tN_n(F)\right)$$

Let $(\pi, V)$ be a representation of $GL_n(F)$. It is called *smooth* (or *algebraic*) if for every $v \in V$, the stabilizer of $v$ in $GL_n(F)$ given by

$$Stab_{GL_n(F)}(v) = \{g \in GL_n(F) \mid \pi(g)v = v\}$$

is open. The representation $\pi$ is said to be *admissible* if $\pi$ is smooth, and the space of vectors fixed by any open compact subgroup of $GL_n(F)$ is finite dimensional. An irreducible admissible representation of $GL_n(F)$ is infinite dimensional if it is not 1 dimensional. If $(\pi, V)$ is a smooth representation of $GL_n(F)$, let $V^*$ denote the space of linear functionals on $V$. We have $GL_n(F)$ acting on $V^*$; the action $\pi^*$ is given by

$$< v, \pi^*(g)f > = < \pi(g^{-1})(v), f > \qquad (g \in GL_n(F), v \in V)$$

Let $\tilde{V}$ denote the subspace of $V^*$ that consists of those linear functionals on $V$ whose stabilizers are open in $GL_n(F)$ under the above action. Then $\tilde{V}$ is stabilized by

$GL_n(F)$ and this representation is called the *contragredient representation* of $(\pi, V)$. On an irreducible admissible representation $\pi$ of $GL_n(F)$ the centre of $GL_n(F)$ acts by a character called the *central character* of $\pi$ denoted by $\omega_\pi$.

On a locally compact group $G$ there exists a left Haar measure that is unique upto a positive real number. Let $dx$ be a left Haar measure on $G$. Uniqueness of Haar measures gives rise to a character $\Delta_G$ of $G$ given by $d(xg) = \Delta_G(g)dx$. This is called the *modular character* of $G$. Further $dx \Delta_G(x)^{-1} dx$ is a right Haar measure on $G$. A left Haar measure is right invariant if and only if the modular character is trivial and in this case $G$ is called *unimodular*. It can be verified that $\Delta_{GL_n(F)} = 1$ and

$$\Delta_{B_n(F)}(diag(a_1, a_2, \dots, a_n)) = |a_1|_F^{1-n}|a_2|_F^{3-n} \cdots |a_n|^{n-1}$$

Suppose $B$ is a closed subgroup of $G$. Let $(\rho, W)$ be a smooth representation of $B$. We get a (smooth) representation of $G$ out of this, which is the *(normalized) induced representation* $I_B^G(\rho)$ and this acts by right translation on the space

$$\left\{ f : G \to W \;\middle|\; \begin{array}{l} f(bg) = \left(\frac{\Delta_G(b)}{\Delta_B(b)}\right)^{1/2}\rho(b)f(g) \text{ for } b \in B, g \in G \\ f(hu) = f(h) \text{ for } u \in U_f \text{ an open set in } G \end{array} \right\}$$

We also have the notion of a *compactly induced representation*, denoted by $ind_B^G(\rho)$, whose space is

$$\{f \in I_B^G(\rho) \mid f \text{ has compact support modulo } B\}$$

Let us denote the corresponding non-normalized inductions by *Ind* and *ind* respectively.

Now we can describe the concept of *parabolic induction* for $GL_n(F)$. For a partition $n = n_1 + n_2 + \dots + n_k$ of $n$ let $P(n_1, n_2, \dots, n_k)$ be the *standard parabolic* *subgroup* given by block upper triangular matrices

$$P(n_1, n_2, \ldots, n_k) = \left\{ \begin{pmatrix} g_1 & * & * & * \\ & g_2 & * & \\ & & \ddots & \\ & & & g_k \end{pmatrix} \;\middle|\; g_i \in GL_{n_i}(F) \right\}$$

A *parabolic subgroup of type* $(n_1, n_2, \ldots, n_k)$ is a conjugate of $P(n_1, n_2, \ldots, n_k)$ in $GL_n(F)$. The *unipotent radical* of the above standard parabolic subgroup is its subgroup with $g_i = 1_{n_i}$. The *Levi subgroup* of $P(n_1, n_2, \ldots, n_k)$ is $GL_{n_1}(F) \times GL_{n_2}(F) \times \ldots \times GL_{n_k}(F)$ embedded diagonally in $GL_n(F)$. Let $\rho$ be a smooth representation of the Levi subgroup. Extend it trivially across the unipotent radical to get a representation of the parabolic subgroup, again denoted by $\rho$. Now consider $I_{P(n_1, \ldots, n_k)}^{GL_n(F)}(\rho)$. This representation is said to be *parabolically induced* from the Levi subgroup to $GL_n(F)$.

Parabolic induction constructs representations of $GL_n(F)$ from representations of its Levi subgroups. There is an adjoint functor to this, called the *Jacquet functor*, which constructs representations of Levi subgroups from representations of $GL_n(F)$. Let $P = MN$ be the Levi decomposition of a parabolic subgroup $P$ of $GL_n(F)$. If $(\rho, V)$ is a representation of a parabolic subgroup of $GL_n(F)$, let

$$V(N) = \{n.v - v \mid n \in N, v \in V\}.$$

The *Jacquet functor* $\rho_N$ of $\rho$ is $V/V(N)$. It can be seen that $\rho_N$ is a representation space for $M$. If $\rho$ is a smooth representation of $GL_n(F)$, its Jacquet functor $\rho_N$ is just that of $\rho|_P$.

For a smooth representation $(\pi, V)$ of $GL_n(F)$, a *matrix coefficient* of $\pi$ is a function on $GL_n(F)$ of the form $<\pi(g)v, \tilde{v}>$ where $v \in V$, $\tilde{v} \in \tilde{V}$, $g \in GL_n(F)$. We l a c

Theorem 2 1 5 Let $(\pi, V)$ be an irreducible admissible representation of $GL_n(F)$ Then the following are equivalent

(i) A matrix coefficient of $\pi$ is compactly supported modulo the centre

(ii) Every matrix coefficient of $\pi$ is compactly supported modulo the centre

(iii) The Jacquet functors of $\pi$ (for all proper parabolic subgroups) are zero

(iv) The representation $\pi$ does not occur as a subquotient of any representation parabolically induced from any proper parabolic subgroup

A representation satisfying any one of the conditions of theorem 2 1 5 is called a *supercuspidal representation* Later in this section we will state yet another characterization of a supercuspidal representation due to Gelfand and Kazhdan in terms of the *Bernstein Zelevinsky derivatives* (Proposition 2 1 7)

A representation $\pi$ is said to be *essentially square integrable* if there is a character $\chi$ of $F$ such that $| < \pi(g)v, v > |^2 \chi(det\ g)$ is a function on $Z(F)\backslash GL_n(F)$ for every matrix coefficient of $\pi$ and the function is integrable on $GL_n(F)$ modulo its centre $Z(F)$ If $\chi$ can be taken to be trivial $\pi$ is said to be *square integrable* or we say that it is in the *discrete series* for $GL_n(F)$

The next theorem says when a representation parabolically induced from a supercuspidal representation is reducible

Theorem 2 1 6 Let $P = P(n_1, n_2, \ldots, n_k) = MN$ be a standard parabolic subgroup of $GL_n(F)$ Let $\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k$ be an irreducible representation of $M$ with every $\rho_i$ a supercuspidal representation of $GL_{n_i}(F)$ The parabolically induced representation $I_P^{GL_n(F)}(\rho)$ is reducible if and only if there exist $1 \quad i, j < k$ with $\neq$

$n_1 = n_2$ and $\rho_1 \sim \rho_2 |\ |_F$

The supercuspidals are the building blocks of all irreducible admissible representations of $GL_n(F)$. We do not intend to state the precise theorems in this regard since in this dissertation we deal only with the supercuspidal representations and the representations parabolically induced from a Levi subgroup of type $(n_1, n_2)$ where the representation of the Levi subgroup is given in terms of two supercuspidals. Suffices to say, for a feel of completeness, that all essentially square integrable representations can be obtained as (a unique) quotient of a representation parabolically induced from the supercuspidals and any irreducible admissible representation of $GL_n(F)$ is obtained the same way out of essentially square integrable representations.

We say that the representation $(\pi, V)$ of $GL_n(F)$ is *unitary* if $V$ has an inner product which is $GL_n(F)$ invariant. A discrete series representation is unitary. Also unitary supercuspidals belong to the discrete series.

Now fix a non trivial additive character $\psi$ of $F$. This gives rise to a character of $N_n(F)$, again denoted by $\psi$, given by

$$\psi((n_{ij})) = \psi(n_{12} + n_{23} + \cdots + n_{n-1\,n})$$

An irreducible admissible representation $(\pi, V)$ of $GL_n(F)$ is said to be *generic* if there exists a non zero linear functional $\Lambda$ on $V$ such that

$$\Lambda(\pi(n)v) = \psi(n)\Lambda(v)$$

for all $v \in V$ and $n \in N_n(F)$. Such a $\Lambda$ is called a *Whittaker functional*. If $\pi$ is generic then the representation space $V$ may be realized on a certain space of smooth functions $\mathcal{W}(\pi, \psi)$ given by

$$\mathcal{W}(\pi, \psi) = \{W : GL_n(F) \to \mathbb{C} \mid W(ng) = \psi(n)W(g),\ n \in N_n(F),\ g \in GL_n(F)\}$$

and the action of $GL_n(F)$ on this space is by right translations. Such a realization is called a *Whittaker model* for $\pi$. The notion of $\pi$ being generic does not depend on the additive character that we choose. The space of Whittaker functionals on an irreducible admissible representation of $GL_n(F)$ has dimension at most one. This multiplicity one result is due to Shalika. Another result due to Zelevinsky states that $\pi$ is generic if and only if it is irreducibly induced from essentially square integrable representations.

We say that $\pi$ is *tempered* if it is irreducibly induced from a discrete series representation.

Let $P_n(F)$ be the *mirabolic subgroup* of $GL_n(F)$ which consists of those $g$ in $GL_n(F)$ with $\eta g = g$ where $\eta = (0, 0, \ldots, 1)$. Let

$$\mathcal{K}(\pi, \psi) = \{W|_{P_n(F)} \mid W \in \mathcal{W}(\pi, \psi)\}$$

and let $P_n(F)$ act on this space by right translations. This is a model for $\pi$ since the map $W \to W|_{P_n(F)}$ is an injection. This realization is called the *Kirillov model* for $\pi$. The space of $ind_{N_n(F)}^{P_n(F)}(\psi)$, say $\mathcal{K}_0(\psi)$, is contained in $\mathcal{K}(\pi, \psi)$ and $\mathcal{K}_0(\psi) = \mathcal{K}(\pi, \psi)$ precisely when $\pi$ is supercuspidal (see proposition 2.1.7 and the discussion preceding that).

We end this section with a brief overview of Bernstein Zelevinsky derivatives.

We have $GL_n(F)$, its mirabolic subgroup $P_n(F)$ and we denote by $U_n(F)$ the unipotent radical of $P_n(F)$. A nontrivial additive character $\psi$ of $F$ gives a character of $U\ (F)$ by $\psi\ ((u\ ))$ $\psi\ (u\ )$ Let *Alg G* denote the category of smooth representations of $G$ for an $l$ group $G$ We have four important functors denoted by $\Psi$ $\Psi$ $\Phi$ and $\Phi$ The functors $\Psi$ and $\Phi$ are from *Alg* $P\ (F)$ to *Alg* $GL_n\ (F)$ and *Alg* $P\ (F)$ respectively $\Psi$ is the Jacquet functor and $\Phi$ the twisted (b ) Jacquet functor Both are normalized Thus if $(\tau\ V)$ is a smooth representation of $P_n(E)$ consider the space $J(V)$ $\{\ (u)v - v \mid v \in V\ \in U_n\ L)\}$ Now the Jacquet functor is $(\Psi\ (\tau)\ V/J(V))$ where $\Psi\ (\tau)(g)(v\ J\ V) - de\ g\ (\tau(g)v + J(V))$ To construct $\Phi$ consider the space $J_\psi(V) = \{\tau(u)v\ \ \psi(u)v \mid v \in V\ u \subset U_n(E)\}$ Then the $\psi$ twisted Jacquet functor is $(\Phi\ (\tau)\ J_\psi(V))$ and the action is given by $\Phi\ (\tau)(p)(v + J_\psi(V))\ \ |det\ p|_E\ ^2(\tau(p)v + J_\psi(V))$ $\Psi^+$ and $\Phi^+$ are maps respectively from *Alg* $GL_n\ (F$ and *Alg* $P\ (F)$ to *Alg* $P_n(F)$ The functor $\Psi$ is just normalized extension by the trivial representation Thus for a $\sigma \in$ *Alg* $GL_n\ (F)$ $\Psi\ (\sigma) \in$ *Alg* $P_n(F)$ is given by $\Psi\ (\sigma)(g) = |det\ g\ \ \sigma(g)$ $\Phi$ is the functor of normalized compactly supported induction If $\sigma$ is a smooth representation of $P_n\ \ F)$ extend it to a representation of $P\ (F)U_n(F)$ by letting $U\ (F)$ act by the acte The $\Phi\ (\sigma) = nd_P^{P\ F}\ \ _{U\ F}(\ det\ \ \sigma \otimes\ )$ here the nd ct o d o normalized using smooth functions of compact support mod o $P_n\ _1(F)U\ (F)$

These functors are used to define the *derivatives* f a smooth representat o of $P_n(F)$ Define the representations $\tau^{k)} \in$ *Alg* $GL_{n\ k}(F)$ $(k\ \ 1\ 2\ \ n)$ y $\tau^{(k}\ \ \Psi\ (\Phi\ )^{k\ 1}(\tau)$ We call $\tau^{\ k}$ the $k$ th derivative of $\tau$ Now there exists a natural filtration by $P_n(F)$ submodules $0 \subset \tau\ \subset \tau_n\ \subset\ \ \subset \tau_1\ \ \tau$ such that $\tau_k/\tau_{k+}\ \ (\Phi^+)^k\ \ \Psi^+(\tau^{(k)})$ He $\tau_k\ (\Phi\ )^k\ (\Phi\ )\ (\ )$ It follows that if $\tau$ is also rred c ble th t s equ al t to a representat o of the form $(\Phi\ )^k\ \Psi\ (\rho)$ ere $1 < l <$ d $\rho$ a ble representation $GL_{n\ k}\ F)$ The index $k$ and representation $\rho$ are uniquely determined by $\tau$

If $\tau$ is a smooth representation of $GL_n(F)$ set $\tau = \pi \ {}_{F}$ Also set $\pi^{0}$ $\tau$ and $\tau^{k}$ $\tau^{k}$ for $k - 1\ 2$ $n$ Then $\tau^{(k}$ are called the derivatives of $\pi$ We have the following results The first is due to Gelfand and Kazhdan (Theorem 4 4 [6]) and the second due to Bernstein and Zelevinsky (Lemma 4 5 [6])

Proposition 2 1 7 The representation $\pi$ is a supercuspidal representation if and only if $\tau^{k} - 0$ for $0 < k < n$ and $\tau$ $- 1$

Proposition 2 1 8 Let $\pi$ be smooth representations of $GL_n(F)$ $i - 1\ 2$ Let $n -$ $n + n_2$ Let $I(\pi_1\ \tau_2)$ denote the representation of $GL_n(F)$ obtained by normalized parabolic induction from the standard parabolic of type $(n\ \ n_2)$ Then $(I(\pi_1\ \ {}_2))^{k}$ the $k$ th derivative of $I(\pi\ \ \pi_2)$ has a filtration whose successive quotients are $I(\quad \pi^{k}$ for $0 < \ < k$

## 2 2 *Functional equations and the local factors*

Our exposition here is based on [9] [15] and [25] Another general reference is [33]

First we state the main theorem of [25] The *epsilon factor for pairs* (or the *convoluted epsilon factor*) appears here Let $\pi$ and $\tau_2$ be irreducible admissible generic representations of $GL_n(F)$ and $CL_n(F)$ respectively For each pair of Whittaker functions $W \in \mathcal{W}(\pi_1\ \psi)$ and $W_2 \in \mathcal{W}(\tau_2\ \upsilon\ \ ^{1})$ and in the case $n_1 = n_2 = n$ each Schwartz Bruhat function $\Phi \in S(F^n)$ define the local integrals as follows If $\tau_2 < \tau$ for any $0\ \ j < n\ \ \ n_2\ \ 1$ we set

$$\Psi(s, W, W_2) \quad \int_{N \quad F \backslash GL \quad F} \int_{M} W \begin{pmatrix} g & & \\ x & 1 & \\ & & 1 \end{pmatrix} W_2(g) \det g \qquad dx\,dg$$

and f $n$ set

$$\Psi(s, W_1, W_2, \Phi) = \int_{N \quad F \backslash GL \quad F} W(g) W_2(g) \Phi(\eta g) |\det g| \, dg$$

where $\eta$ $(0, 0, \quad 1)$

Theorem 2 2 1 Let $\pi$ and $\pi_2$ be irreducible admissible generic representations of $GL \quad (F)$ and $GL_n \quad (F)$ respectively Let $W \in \mathcal{W}(\pi \quad)$ and $W_2 \in \mathcal{W}(\pi_2 \quad)$ T

( ) The integrals $\Psi(s, W, W_2, \Phi)$ $\Psi(s, W, W \quad)$ converge absolutely for $Re(\ )$ large

(ii) The a e n oral f r c o of M r e selv f $- n_2 - n$ the integral $\Psi(s, W, W_2, \Phi)$ span a fractional ideal of e ring $\mathbb{C}[X, X \quad] \; X \qquad q$ h l generated by a factor $L(s, \pi \times \pi_2)$ that has the form $P(X)$ here $P \in \mathbb{C}[X]$ and $P(0) \quad 1$ If $\quad _2 < r \quad 1$ there s a s lar factor $L( \quad \pi \times \pi_2$ independent of $j$ generating the ideal spanned by the integrals $\Psi(s, W_1, W_2)$ The same results are true for the pair of representations $( \quad \pi_2)$

(iii) Suppose $- n_2 - i$ Then there is a factor $\epsilon(s, \pi \times \pi_2, \psi)$ of the form $q \quad ^m$ such that

$$\frac{\Psi( \qquad W \quad \widetilde{W}_2 \; \Phi)}{L(1 \qquad \pi \times \quad )} \qquad ( \qquad \times \qquad ) \frac{\Psi \quad W \quad W_2 \; \Phi)}{L \quad \pi \times \quad )}$$

Similarly if $n_2 < n - 1$ set $k = n - n_2 - 1 - j$. Then

$$\frac{\Psi_k(1-s,\rho(w_{n,n_2})\widetilde{W},\widetilde{W_2})}{L(1-s,\tilde{\pi}\times\tilde{\pi}_2)} = \omega_{\pi_2}(-1)^{n-1}\epsilon(s,\pi\times\pi_2,\psi)\frac{\Psi(s,W,W_2)}{L(s,\pi\times\pi_2)}$$

and the epsilon factor is independent of $j$. Here $w_{n,n_2} = \begin{pmatrix} 1 & 0 \\ 0 & w_{n-n_2} \end{pmatrix}$ and $w_n$ denote the longest Weyl element in $GL(n)$ with 1 on the anti diagonal. $\Phi$ is the Fourier transform taken with respect to the $\psi$ self dual Haar measure. For $W \in \mathcal{W}(\pi,\psi)$ with $\pi$ a representation of $GL(n)$, $\widetilde{W}(g) = W(w_n\, {}^tg^{-1})$ and then $\widetilde{W} \in \mathcal{W}(\tilde{\pi},\psi^{-1})$. $\rho$ denotes the operator of right translation. Also $q = q_F$.

Remark. Making use of the (Langlands) classification of all irreducible admissible representations of $GL_n(F)$ (alluded to in the previous section), one can define the $L$ function and the $\epsilon$ factor for all the irreducible admissible representations.

We consider a degree 2 extension of $F$, say $E$. Let $\psi_E$ be the additive character of $E$ given by

$$\psi_E(x) = \psi(Tr_{E/F}(\Delta x))$$

where $\Delta$ is an element of $E$ of trace zero. Then $\psi_E$ has trivial restriction to $F$ and any character of $E$ with trivial restriction to $F$ is of this form. Let $< \sigma > = Gal(E/F)$. Let $\Pi_1$ and $\Pi_2$ be irreducible admissible generic representations of $GL_{n_1}(E)$ and $GL_{n_2}(E)$ respectively. We list a set of properties of the epsilon factor for pairs $\epsilon(s,\Pi_1 \times \Pi_2,\psi)$ in the following proposition.

Proposition 2.2.2 (i) $\epsilon(s,\Pi_1^\sigma \times \Pi_2^\sigma,\psi) = \epsilon(s,\Pi_1 \times \Pi_2,\psi)$

(ii) $\epsilon(s,\Pi_1 \times \Pi_2,\psi)\epsilon(1-s,\tilde{\Pi}_1 \times \tilde{\Pi}_2,\psi^{-1}) = 1$

(iii) Let $a \in E$ Let $\psi_a$ be the additive character given by $\psi_a(x) - \psi(ax)$ Then

$$\epsilon(s\ \Pi\ \times \Pi_2\ \psi_a)\quad \omega_{\Pi}^{n}\,\omega_{\Pi}^{n}\,(a)|\,x\,|_{E}^{n\ n_2}\quad {}^{2}\,\epsilon(s\ \Pi_1 \times \Pi_2\ \psi)$$

(iv) For every character $\chi$ of $E^*$ with sufficiently large conductoral exponent we have

$$\epsilon(s\ \chi\Pi\ \times \Pi_2\ \psi)\quad \epsilon(s\ \chi\ \psi)^{n\ n_2-1}\epsilon(s\ \chi\omega_{1}^{n}\,\omega_{1}\ \ \psi)$$

Define

$$\gamma(s\ \Pi_1 \times \Pi_2\ \psi) - \epsilon(s\ \Pi_1 \times \Pi_2\ \psi)\frac{L(1 - s\ \widetilde{\Pi}\ \times \Pi_2)}{L(s\ \Pi_1 \times \Pi_2)}$$

We have

Proposition 2 2 3 (Multiplicativity of $\gamma$ factors) $\Pi$ $I(\Pi_1\ \Pi_2)$ be the representation of $GL\ (E)$ $n - n\ + n_2$ which is parabolically induced (normalized) from $GL_n\ (E) \times$ $GL\ (E)$ Then

$$(s\ \Pi \times \Pi\ \ \psi)\quad \gamma(s\ \Pi\ \times \Pi\ \ \psi)\gamma(s\ \Pi_2 \times \Pi\ \ \psi)$$

and similarly for $\Pi$ Moreover $L(s\ \Pi \times \Pi\ )^{-1}$ divides $[L(s\ \Pi\ \times \Pi\ )L(s\ \Pi_2 \times \Pi\ \ ]$

Now let us introduce the *twisted epsilon factor* We have $E/F$ a quadratic extension of local fields of characteristic zero Let $\Pi$ be an irreducible admissible generic representation of $GL_n(E)$ with a unitary central character $\mathcal{W}(\Pi\ \psi)$ is its $\psi$ Whittaker model For $W \in \mathcal{W}(\Pi\ \psi)$ and a Schwartz Bruhat function $\Phi \in \mathcal{S}(F^n)$ consider the integral

$$\Psi(s\ W\ \Phi) = \int_{N\ (F\ \ GL\ (F}\ W\ (g)\Phi\,(\tau\, g)|det\ g|_{F}^{s}dg$$

where $(0\ 0\ \ \ 0\ 1)$ and $dg$ is the right $GL\ (F)$ invariant measure on the quotient space Then we have (t a ı theorem [15])

Theorem 2.2.4 (i) For each $W \in \mathcal{W}(\Pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$, the integral $\Psi(s, W, \Phi)$ is absolutely convergent for a large $Re(s)$ to a rational function of $X \quad q$

(ii) There exists a polynomial $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that the integrals $\Psi(s, W, \Phi)$ span the fractional ideal $L(s, r(\Pi))\mathbb{C}[X, X \ ]$ of the ring $\mathbb{C}[X, X^{-1}]$ where $L(s, r(\Pi)) \quad P(X)$

(iii) There exists an integer $m(\Pi, \psi)$ and a non zero complex number $c(\Pi, \psi)$ such that

$$\frac{\Psi(1-s, W, \Phi)}{L(1-s, r(\Pi))} = \omega_{\Pi}(-1)^{n-1}\epsilon(s, \ (\Pi), \psi)\frac{\Psi(s, W, \Phi)}{L(s, r(\Pi))}$$

for all $W \in \mathcal{W}(\Pi, \psi)$, $\Phi \in \mathcal{S}(F \ )$. Here $\omega_{\Pi}$ is the central character of $\Pi$, $\Pi$ the contragredient of $\Pi$ and we put $\epsilon(s, r(\Pi), \psi) = c(\Pi, \psi)X^{m(\Pi, \psi)}$, $W(g) \quad W(\omega \ g \ )$, $\Phi(x) = \int_F \Phi(y)\psi(\sum x \ y \ )dy$, $\omega \in GL \ (F)$ is the longest Weyl element whose non zero entries are 1 located on the anti diagonal. The Fourier transform of $\Phi$ is taken with respect to the self dual Haar measure for

Remark. In fact there exists some small constant $\epsilon$, $\epsilon > 0$ such that the integral $\Psi(s, W, \Phi)$ converges absolutely uniformly in compact subsets for $Re(s) > 1 -$ (Proposition [17])

We record some useful properties of the twisted epsilon factor here

Proposition 2.2.5 (i) $\epsilon(s, r(\Pi^{\sigma}), \psi^{-1}) = \omega_{\Pi}(-1)\epsilon(s, r(\Pi), \psi)$ where $\sigma$ is the non trivial element in $Gal(E/F)$ and $\Pi \ (g) = \Pi(g^{\sigma})$

(ii) $\epsilon(s, \ (\Pi), \psi)\epsilon(1-s, r(\Pi), \psi^{-1}) = 1$

(iii) Let $a \in F$. Let $\psi_a$ be the character of $E$ given by $\psi_a(x) = \psi(ax)$. Then

$$\epsilon(s, r(\Pi), \psi_a \quad \omega(\ )\ a^{n} \quad {}^{2}\ (\quad \Pi), \psi)$$

It will be nice to have an analogue of proposition 2 2 3 But this is conjectural (see [15])

Proposition 2 2 6 (Conjectural relations) Let $\Pi = I(\Pi_1 \ \Pi_2)$ as in proposition 2 2 3 Then (i) $L(s \ r(\Pi)) = L(s \ r(\Pi_1))L(s \ r(\Pi_2))L(s \ \Pi_1 \times \Pi_2^\sigma)$
(ii) $\epsilon(s \ r(\Pi) \ \psi) = \epsilon(s \ r(\Pi_1) \ \psi)\epsilon(s \ r(\Pi_2) \ \psi)\epsilon(s \ \Pi_1 \times \Pi_2^\sigma \ \psi_E)$

## 2 3 Functoriality

We have [4] [11] and [23] as the references

Let $\Pi$ be an irreducible admissible representation of $GL_n(E)$ such that $\Pi \sim \Pi^\sigma$ where $\Pi^\sigma(g) = \Pi(g^\sigma)$ We also assume that $\Pi$ is generic Then $\Pi$ is said to be a *base change lift* of a representation $\pi$ of $GL_n(F)$ if for $g \in GL_n(E)$ such that $N_{E/F}(g)$ is regular

$$trace(\Pi(g)I_\sigma) = trace(\pi(N_{E/F}(g)))$$

where $I_\sigma$ is defined on $\mathcal{W}(\Pi \ \psi_E)$ by $I_\sigma(W) = W^\sigma$ Here $\psi_E(x) = \psi(Tr_{E/F}(x))$

Remark Using the Langlands classification the base change lift is defined for all the irreducible admissible representations of $GL_n(F)$ (though not in terms of a character identity as above)

The main properties of the base change lift are summarized in the next proposition

Proposition 2 3 1 (i) Any $\pi$ has a lift $\Pi$ and if $\Pi$ is a lift then $\Pi \sim \Pi^\sigma$
(ii) Suppose $\Pi \sim \Pi^\sigma$ Then there exists $\pi$ such that $\pi$ lifts to $\Pi$
(iii) $\omega_\Pi = \omega_\pi \circ N_{E/F}$ if $\Pi$ is a lift of $\pi$
(iv) If $\pi$ lifts to $\Pi$ then $\chi \otimes \pi$ lifts to $(\chi \circ N_{E/F}) \otimes \Pi$
(v) If $\tau$ lifts to $\Pi$ then lifts to $\Pi$

We also have the following proposition with local factor identities

Proposition 2 3 2 Let $\pi_1$ and $\pi_2$ be irreducible admissible representations of $GL\ (F)$ and $GL_n\ (F)$ respectively Let these base change to $\Pi_1$ and $\Pi_2$ representation of $GL_n\ (E)$ $i = 1\ 2$ Then we have the identities

(i) $L(s\ \Pi_1 \times \Pi_2) = L(s\ \ \times \pi_2)L(s\ \pi\ \times \pi_2 \times \omega_E\ )$

(ii) $\epsilon(s\ \Pi\ \times \Pi_2\ \psi_E) = \lambda(E/F\ \psi_E)^{n\ n}\ \epsilon(s\ \pi_1 \times \pi_2\ \psi\ )\epsilon(\ \pi_1 \times \pi_2 \times \omega_{E\ F}\ \psi\ )$

Here $\lambda(E/F\ \psi_E)$ is the *Langlands factor* given by

$$\lambda(E/F\ \psi_E) = \frac{\epsilon(s\ 1_F\ \psi\ )\epsilon(s\ \omega_{E\ F}\ \psi\ )}{\epsilon(s\ 1\ \ \psi_E)}$$

Under the *local Langlands correspondence* the base change map from $GL\ (F)$ to $GL_n(E)$ corresponds to the restriction map on the Galois group side There is a functorial lift that corresponds to induction on the Galois side which is called *automorphic induction* Thus automorphic induction takes a representation $\Pi$ of $GL_n(E)$ to a representation of $GL_{2n}(F)$ We do not get into the relevant character identity but we enlist the main properties of this lift As in [23] we assume $\Pi$ to be tempered

Proposition 2 3 3 ( ) Any $\Pi$ lifts to a $\pi$ with $\pi \sim \pi \otimes \omega_{E\ F}$

(ii) Suppose $\pi \sim \pi \otimes \omega_{E\ F}$ then there is a $\Pi$ such that $\pi$ is a lift of $\Pi$

(i ) If $\Pi$ lifts to $\pi$ then $\omega_\pi = \omega_{E\ F}\omega_{\Pi\ F}$

(i ) Let $\chi$ be a character of $F$ Suppose $\pi$ is a lift of $\Pi$ Then $(\chi \circ N_{E\ F}) \otimes \Pi$ lifts to $\chi \otimes \pi$

(v) If $\Pi$ lifts to $\pi$ then $\Pi$ lifts to $\pi$

(vi) If $\Pi$ lifts to $\pi$ then $\Pi^\sigma$ also lifts to $\pi$

The next proposition is similar to proposition 2.3.2.

Proposition 2.3.4. Let $\Pi_1$ and $\Pi_2$ be irreducible tempered representations of $GL_n(E)$ and let $\pi_1$ and $\pi_2$ be their lifts under automorphic induction. Then

(i) $L(s, \pi_1 \times \pi_2) = L(s, \Pi_1 \times \Pi_2) L(s, \Pi_1 \times \Pi_2^\sigma)$

(ii) $\epsilon(s, \pi_1 \times \pi_2, \psi) = \lambda(E/F, \psi_E) \epsilon(s, \Pi_1 \times \Pi_2, \psi_E) \epsilon(s, \Pi_1 \times \Pi_2^\sigma, \psi_E)$

Let $J_n$ be the $n \times n$ matrix whose $(i, j)$ entry is $(-1)^{n-1}\delta_{i,n+1-j}$. Let $\tau$ denote the involution of $GL_n(E)$ given by $g^\tau = J_n\, {}^t g^{-\sigma} J_n^{-1}$. Let

$$U(n, E/F) = \{g \in GL_n(E) \mid g^\tau = g\}$$

Now we take $n = 2$. Call $J = J_2$. We describe the base change map from $U(2, E/F)$ to $GL_2(E)$ as in [11]. The image of the base change map from the class of admissible representations of $U(2, E/F)$ to the class of admissible representations of $GL_2(E)$ consists of $\tau$-invariant $\Pi$. The central character of any irreducible admissible representation $\Pi$ of $GL_2(E)$ which is in the image of the base change map is trivial on $F^\times$. If $\Pi$ is $\tau$-invariant and $\omega_\Pi|_{F^\times} = 1$, then $\Pi$ is obtained as the base change of a unique $L$-packet of $U(2, E/F)$. This $L$-packet consists of one or two irreducible admissible representations of $U(2, E/F)$. If $\Pi$ is an admissible representation of $GL_2(E)$ such that $\Pi \sim \Pi^\tau$, then take an intertwining operator between the spaces of $\Pi$ and $\Pi^\tau$ and use this operator to extend $\Pi$ to the semi-direct product $GL_2(E) \times Gal(E/F)$ where $Gal(E/F)$ acts on $GL_2(E)$ by $\sigma \cdot g = g^\tau$. Let $\chi_\Pi$ denote the character of this extended representation. There are precisely two base change maps, stable and unstable, from the class of admissible representations of $U(2, E/F)$ to the class of admissible representations of $GL_2(E)$. Let $\omega_{E/F}$ be an extension of $\omega_F$ to $E^\times$. We say that Π is a stable base change of a representation π of $U(2\ E/F)$ if

$$\chi_{\Pi}\ (g) = \chi_{\ \pi\}}(gg\ )$$

whenever $g$ is such that $gg$ is regular in $U(2\ E/F)$ Here $\{\tau\}$ is the $L$ packet of and $\chi_{\{}\ = \chi\ + \chi_{\pi}$ The character $\chi_{\ \}}$ depends only on the conjugacy class of $gg$ in $GL_2(E)$ Further Π is said to be an unstable base change lift of $\tau$ if

$$\chi_{\Pi}\ (g) - \tilde{\omega}\ (det\ g)\chi_{\{\ \}}(gg\ )$$

for all $g \in GL_2(E)$ with $gg$ regular in $U(2\ E/F)$

We will need the following result (see p 161 [13] p 717 [11])

Proposition 2 3 5 (i) The principal series representation $\Pi(\chi\ \chi^{\sigma})$ of $GL_2(E)$ is in the image of both the stable and the unstable base change maps

(ii) The principal series representation $\Pi(\chi\ \chi_2)$ with $\chi \neq \chi_2$ and $\chi|_F - 1\ (i - 1\ 2$ is in the image of the unstable base change map and it is not obtained by the stable lifting

(iii) The special representation $\Sigma(\chi\ \chi_{E}\ ^2)$ is obtained through the unstable lifting precisely when $\chi_F\ \omega$

## 2 4 *Quasi invariant measures*

We recall the following from [35] (appendix pp 4 4 477) Note that we take the left cosets and consider the right measure whereas it is the other way round in our reference

Let $C$ be a locally compact group and $H$ a closed subgroup Let $\rho$ be a strictly positive Borel function on $G$ bounded above and bounded on compact subsets such that for every $h \in H$ we have

$$\rho(hg) = \frac{\Delta_G(h)}{\Delta_H(h)}\rho(g) \quad (g \in G)$$

where $\Delta_G$ and $\Delta_H$ are modular functions for $G$ and $H$ respectively. A function with these properties is called a *rho function*. If we fix a rho function $\rho$ associated to $H$ we have a *quasi-invariant measure* $\mu_\rho$ defined by

$$\int_G f(x)\rho(x)\,dx = \int_{H\backslash G}\int_H f(hx)\,dh\,d\mu_\rho(x)$$

where $f$ is continuous with compact support.

Suppose $H_1$ and $H_2$ are closed subgroups of $G$ and $H_1 \subset H_2$. We have the lemma (Lemma A.1.2 [35])

**Lemma 2.4.1** Suppose $H_1\backslash H_2$ admits a positive $H_2$ right invariant measure $\nu_2$. Let $\rho_2$ be a rho function on $G$ for $H_2$ (and hence also for $H_1$ since $\Delta_{H_2}|_{H_1} = \Delta_{H_1}$). Let $\mu_1$ and $\mu_2$ be the associated quasi-invariant measures on $H_1\backslash G$ and $H_2\backslash G$. Then for a suitable normalization of Haar measures we have

$$\int_{H_1\backslash G} f(x)\,d\mu_1(x) = \int_{H_2\backslash G} d\mu_2(x)\int_{H_1\backslash H_2} f(hx)\,d\nu_2(h)$$

where $f$ is continuous with compact support modulo $H_1$.

# 3 DISTINGUISHEDNESS FOR $GL(2)$

## 3.1 *Distinguishedness for quadratic extensions*

First we intend to summarize certain results that we will need in the sequel. Thus let Π be an irreducible admissible representation of $GL_n(F)$. We start with the following multiplicity one result due to Flicker (Proposition 11, [13]).

Theorem 3.1.1 $Hom_{GL_n(F)}(\Pi, 1)$, the space of $GL_n(F)$ invariant linear forms on the space of Π, has dimension at most one.

Recall that Π is said to be distinguished with respect to $GL_n(F)$, or simply distinguished, when $Hom_{GL_n(F)}(\Pi, 1)$ is not zero. Note that if Π is distinguished, then the central character $\omega$ of Π has trivial restriction to $F$. Another necessary condition for distinguishedness is the content of the next theorem (Proposition 12, [13]).

Theorem 3.1.2 Suppose Π is distinguished with respect to $GL_n(F)$. Then $\Pi \cong \Pi$.

Conjecturally the necessary conditions mentioned above are almost sufficient too. The precise conjecture, due to Jacquet, is as follows.

Conjecture 3.1.3 Let Π be a representation of $GL_n(E)$ with $\omega$ 1. If even then $\Pi \sim \Pi$ if and only if Π is distinguished or $\omega_E$ distinguished. If $n$ is odd then $\Pi \simeq \Pi$ if and only if Π is distinguished.

Theorem 1.3.3 says that the conjecture is true when $n = 2$. It is verified in more general cases too [30]. There is a stronger conjecture due to Flicker and Rallis [13] which says that a distinguished representation corresponds to the unstable (resp. stable) base change lift from $U(n, E/F)$ when $n$ is even (resp. odd). Theorem 1.3.5 verifies this when $n = 2$.

Next we state Hakim's $\gamma$ factor criterion for distinguishedness for $GL(2)$ (Theorem 4.1 [20]).

Theorem 3.1.4. Let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ whose central character restricts trivially to $F$. Then $\Pi$ is distinguished if and only if $\gamma(\frac{1}{2}, \Pi\otimes\lambda^{-1}, \psi) = 1$ for all characters $\lambda$ of $E$ such that $\lambda|_F = 1$.

Let us give a quick proof of Hakim's result. The functional equation (Theorem 2.2.1 (ii)) at $s = \frac{1}{2}$ is given by

$$\int_{E} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\lambda(g)\,d^{\times}g = \gamma(\tfrac{1}{2}, \Pi\otimes\lambda, \psi)\int_{E} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\lambda^{-1}(g)\,d^{\times}g$$

Therefore $\gamma(\frac{1}{2}, \Pi\otimes\lambda, \psi) = 1$

$$\begin{aligned}
&\Leftrightarrow \int_F W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\lambda(g)\,d^{\times}g = \int_E W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\lambda^{-1}(g)\,d^{\times}g \\
&\Leftrightarrow \int_{E/F}\int_F W\begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix}\lambda(gh)\,d^{\times}h\,d^{\times}g \\
&\qquad \int_{E}\int_F W\begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix}\lambda\,d^{\times}h\,d^{\times}g
\end{aligned}$$

$$\Leftrightarrow \int_{E^\times/F^\times} \lambda(g) \left( \int_{F^\times} \Pi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \widetilde{W} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h \right) d^\times g =$$

$$\int_{E^\times/F^\times} \lambda^{-1}(g) \left( \int_{F^\times} \Pi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h \right) d^\times g$$

$$\Leftrightarrow \int_{E^\times/F^\times} \lambda(g) \left( \int_{F^\times} \widetilde{\left( \Pi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W \right)} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h \right) d^\times g =$$

$$\int_{E^\times/F^\times} \lambda^{-1}(g) \left( \int_{F^\times} \Pi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h \right) d^\times g$$

$$\Leftrightarrow \int_{F^\times} \widetilde{W} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h = \int_{F^\times} W \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} d^\times h$$

$\Leftrightarrow$ $\Pi$ is distinguished (by Theorem 1.3.7)

Note that theorem 1.3.7 holds true for special representations of $GL_2(E)$ also though the statement does not cover that case.

The next result explicitly gives all the non supercuspidal distinguished representations of $GL_2(E)$ (Proposition B17, [16]).

**Theorem 3.1.5** The principal series representation $\Pi(\chi, \chi^{\sigma})$ of $GL_2(E)$ is distinguished (and $\omega_{E/F}$ distinguished). The principal series representation $\Pi(\chi_1, \chi_2)$, $\chi_1 \neq \chi_2$ is distinguished precisely when $\chi_i|_{F^\times} = 1$ $(i = 1, 2)$. The special representation $\Sigma(\chi |\ |_E^{2}, \chi |\ |_F^{2})$ is distinguished precisely when $\chi|_{F^\times} = \omega_{E/F}$.

Let us deduce two corollaries here. The first one is a well known result due to Frohlich and Queyrut (Theorem 3.18) and the second is from [3].

**Corollary 3.1.6** Let $\chi$ be a character of $E^\times$ with $\chi|_{F^\times} = 1$. Then $\epsilon(\frac{1}{2}, \chi, \psi) = 1$

Corollary 3 1 7 Let $\chi_1$ and $\chi_2$ be two characters of $E$ such that $\chi\ _F - \lambda_2|_F \neq 1$ and $\gamma(\tfrac{1}{2}\ \chi_1\lambda\ \psi) - \gamma(_2\ \lambda_2\lambda\ \psi)$ for all $\lambda$ such that $\lambda|_F\quad 1$ Then $\lambda\quad \chi$

To prove the first corollary let $\chi$ be a character of $E$ such that $\chi|\quad 1$ If $\chi - 1$ the result follows [illegible] all If $\chi \neq 1$ consider the principal series representation $\Pi\quad \Pi(\chi\quad 1)$ This is distinguished by theorem 3 1 5 So by theorem 3 1 4 $\gamma(_2\ \Pi \otimes \chi\ \psi) - 1$ But $\gamma(_2\ \Pi \otimes \chi\ \psi) - \gamma(_2\ \chi\ \psi)\ (_2\ 1\ \psi) = \ (_2\ \lambda\ \psi)$ Since $\chi$ $\chi\quad - \chi^\sigma$ and therefore $c(_2\ \chi\ \psi) - \gamma(_2\ \chi\ \psi)$ and hence the result One only has to notice that the proof of theorem 3 1 5 (as in [16]) does not make use of the result of Frohlich and Queyrut

To deduce the second corollary we are given that $\gamma(_2\ \chi\ \lambda\ \psi)\quad \gamma(_2\ \chi_2\lambda\ \psi)$ for all $\lambda$ such that $\lambda\quad 1$ Consider the principal series representation $\Pi = \Pi(\chi\quad {\chi_2}^\sigma)$ No

$$
\begin{aligned}
\gamma(\tfrac{1}{2}\ \Pi \otimes \lambda\ \psi) &= \gamma(\tfrac{1}{2}\ \chi\ \lambda\ \psi)\gamma(\quad {\chi_2}^\sigma\lambda\ \psi) \\
&= \gamma(\tfrac{1}{2}\ \chi\ \lambda\ \psi)\gamma(_2\ \lambda_2\ \lambda\quad \psi\ ) \\
&\quad 1
\end{aligned}
$$

for all $\lambda$ such that $\lambda\quad 1$ since the gamma factor is $\sigma$ invariant $\lambda\quad \lambda$ and $\psi\quad - \psi^\sigma$ (see proposition 2 2 2) Therefore by theorem 3 1 4 $\Pi$ is distinguished Since $\chi \neq 1$ on $F$ by our assumption we get $\chi\ = \lambda_2$ by theorem 3 1 5

We end this section with two more statements The first is a formula for characters of $GL(2)$ due to Tunnell [34] proved by Saito in complete generality [32] and the second is a corollary to Saito s proof of Tunnell s formula (Corollary 2 4 [32 )

Theorem 3 1 8 Fix a[illegible]bedding of $E$ in $GL_2(F)$ Let $\pi$ be an irreducible admis sible representation of $GL(\ F)$ and $\chi_\pi$ its character Let $\Pi$ be the base change lift of to $E$ Then

$$\lambda_{E/F} = \sum_{\lambda} \frac{1+\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1)}{2}\lambda$$

where the summation on the right is by partial sums over all characters of $E$ of conductor exponent $< n$

Theorem 3 1 9 Let $\pi$ be a supercuspidal representation of $GL_2(F)$ with central character $\omega_\pi$ and let $\Pi$ be the base change lift of $\pi$ to $GL_2(E)$ Then for characters $\lambda$ of $E$ which satisfy $\lambda|_F = \omega_\pi\omega_{E/F}$ $\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1)$ is independent of $\lambda$

## 3 2 Proof of theorem 1 3 1

We prove theorem 1 3 1 through a series of propositions Our starting point is theorem 3 1 9

First we claim that the value of $\epsilon(\Pi\otimes\lambda^{-1},\psi_F)\lambda(-1)$ in theorem 3 1 9 is precisely 1 To prove this write $\omega_\Pi = \mu_1\mu_2$ where $\mu_1$ and $\mu_2$ are characters of $F$ such that $\mu_i|_F = \omega_{E/F}$ $(i = 1, 2)$ Then for characters $\lambda$ of $E$ with very small conductors which satisfy $\lambda|_F = \omega_\pi\omega_{E/F}$ we have

$$\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1) = \epsilon(\mu_1\lambda^{-1},\psi_E)\epsilon(\mu_2\lambda^{-1},\psi_F)\lambda(-1)$$

by a result of Jacquet and Langlands (Proposition 3 8, p 116, [24] or see proposition 2 2 2 (v)) Note that $\mu_i\lambda^{-1}|_F = 1$ for $i = 1, 2$ Thus by corollary 3 1 6 $\epsilon(\mu_i\lambda^{-1},\psi_E) = \mu_i\lambda^{-1}(\Delta)$ where $\Delta$ is any trace zero element of $E$ I

$$\begin{aligned}\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1) &= \mu_1\lambda^{-1}(\Delta)\mu_2\lambda^{-1}(\Delta)\lambda(-1)\\ &= \mu_1\mu_2(\Delta)\lambda^{-1}(-\Delta^2)\\ &= \omega_\Pi(\Delta)\lambda^{-1}(N_{E/F}\Delta))\\ &= \omega_\Pi(\Delta)\omega_\pi^{-1}(N_{E/F}(\Delta))\\ &= 1\end{aligned}$$

whenever $\lambda|_{F^\times} = \omega_\pi\omega_{E/F}$ and the conductor of $\lambda$ is sufficiently small. This proves our claim.

If $\Pi$ is principal or special then $\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1) = 1$ whenever $\lambda|_F = \omega_\pi\omega_{E/F}$ can be proved by a direct epsilon factor computation and the condition that it be independent of $\lambda$ is not needed. For then either $\Pi = \Pi(\chi_1,\chi_2)$ where $\chi_1$ and $\chi_2$ are characters of $F$ such that $\chi_i = \lambda$ $(i = 1, 2)$ or $\Pi = \Sigma(\chi_1,\chi_2)$ where $\chi_1\chi_2^{-1} = |\;|_E$ and if $\chi_1 = \chi|\;|_E^{1/2}$, $\chi_2 = \chi|\;|_E^{-1/2}$ then $\chi = \mu\circ N_{E/F}$ for a character $\mu$ of $F^\times$. Now consider $\epsilon(\Pi\otimes\lambda^{-1},\psi_E)\lambda(-1)$ for characters $\lambda$ of $E$ with $\lambda|_F = \omega_\pi\omega_{E/F}$. The condition on $\lambda$ means that $\lambda\lambda^{\sigma} = \omega_\Pi$. If $\Pi$ is the principal series representation considered before or the special representation with $\chi\lambda^{-1}$ ramified then the $GL(2)$ $\epsilon$-factor factorizes into the $GL(1)$ factors as follows

$$\epsilon(\Pi\otimes\lambda^{-1},\psi_E) = \epsilon(\chi_1\lambda^{-1},\psi_E)\epsilon(\chi_2\lambda^{-1},\psi_E)$$

We have $\chi_2\lambda^{-1} = \chi_1^{-1}\lambda^{\sigma}$ and hence

$$\epsilon(\chi_2\lambda^{-1},\psi_E) = \epsilon(\chi_1^{-1}\lambda^{\sigma},\psi_E) = \epsilon(\chi_1^{-\sigma}\lambda,\psi_E) = \epsilon(\chi_1^{-1}\lambda,\psi_E)$$

…e $\chi$ $\lambda^{\sigma}$. Therefore

$$\begin{aligned}\epsilon(\Pi\otimes\lambda \quad \psi_E)\lambda(\ 1) &\quad \epsilon(\chi\ \lambda \quad \psi_F)\epsilon(\lambda \quad \lambda\ \psi_E)\lambda(-1)\\ &- \chi_1\lambda^{\ 1}(\ 1)\lambda(-1)\\ &\quad \chi\ (\ 1)\\ &\quad 1\end{aligned}$$

If $\Pi$ $\Sigma(\chi \quad \lambda_2)$ th $\lambda\lambda$ unramified then $E/F$ s necessarily unramified since $\lambda\lambda \ |$ $\omega$ in this situation. So we can take $\varpi_E$ to be $\varpi_F$ itself. Note that $\omega$ $(\varpi_F)$ 1. Now we have the factorization (see p. 109 [24])

$$\epsilon(\Pi\otimes\lambda \quad \psi_E) \quad \epsilon(\lambda\ \lambda \quad \psi_E)\epsilon(\chi_2\lambda \quad \psi_E)\frac{L(\frac{1}{2}\ \chi \quad \lambda)}{L(\frac{1}{2}\ \chi_2\lambda \quad )}$$

Therefore

$$\begin{aligned}\epsilon(\Pi\otimes\lambda^{\ 1}\ \psi_E)\lambda(\ 1) &\quad \epsilon(\chi_1\lambda^{\ 1}\ \psi_F)\epsilon(\chi_2\lambda^{\ 1}\ \psi_E)\frac{\omega_{E/F}(\varpi\ )}{\omega \quad {}_F\ \varpi}\lambda(-1)\\ &\quad \epsilon(\chi_1\lambda^{\ 1}\ \psi_E)\epsilon(\chi_2\lambda \quad \psi_F)\lambda(\ 1)\\ &- \chi\lambda \quad (\ 1)\lambda(\ 1) \quad \chi(-1)\\ &\quad 1\end{aligned}$$

If $\Pi$ s a base change lift of a representation $\tau$ of $GL_2(F)$ then $\Pi \simeq \Pi$ and therefore $\Pi\otimes\lambda$ $(\Pi\otimes\lambda \quad )$ when $\lambda|$ $\omega_\pi\omega_E$. Hence

$$L(\frac{1}{2}\ \Pi\otimes\lambda) = L(\frac{1}{2}\ \Pi\otimes\lambda \quad )$$

Thus it follows that in our situation

$$\epsilon(\Pi\otimes\lambda \quad \psi_E)\lambda(\ 1) - \gamma(\Pi\otimes\lambda \quad \psi_E)\lambda(\ 1)$$

Now if $\Pi$ is an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi$ $\ \circ N_{E/F}$ and if $\Pi \cong \Pi$ then $\Pi$ a base change lift of a representation $\tau$ of $GL_2(F)$ and $\omega$ can be the $\ \circ$ $\omega_F$. Thus from the preceding discussion get

**Proposition 3.2.1** Let $\mu$ be a character of $F^\times$ and let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$ such that $\Pi \cong \Pi^\sigma$. Then $\gamma(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1) = 1$ for all characters $\lambda$ of $E^\times$ with $\lambda|_{F^\times} = \mu$ or $\gamma(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1) = -1$ for all characters $\lambda$ of $E^\times$ with $\lambda|_{F^\times} = \mu\omega_{E/F}$.

Next we prove

**Proposition 3.2.2** Let $\mu$ be a character of $F^\times$ and let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$. Then $\Pi$ is $\mu$-distinguished with respect to $GL_2(F)$ if and only if $\gamma(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1) = 1$ for all characters $\lambda$ of $E^\times$ with $\lambda|_{F^\times} = \mu$.

This is immediate from theorem 3.1.4. Suppose $\Pi$ is an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$. Let $\mu$ be a character of $E^\times$ such that $\mu|_{F^\times} = \mu$. Now $\Pi$ is $\mu$-distinguished if and only if $\Pi \otimes \mu^{-1}$ is distinguished. Note that $\omega_{\Pi \otimes \mu^{-1}}|_{F^\times} = 1$. Thus by theorem 3.1.4 $\Pi$ is distinguished if and only if $\gamma(\Pi \otimes \mu^{-1}\lambda^{-1}, (\psi_F)_\Delta) = 1$ for all characters $\lambda$ of $E^\times$ which satisfy $\lambda|_{F^\times} = 1$, i.e. $\Pi$ is $\mu$-distinguished if and only if $\gamma(\Pi \otimes \lambda^{-1}, (\psi_F)_\Delta) = 1$ for all characters $\lambda$ of $E^\times$ with $\lambda|_{F^\times} = \mu$. Also

$$\begin{aligned}
\gamma(\Pi \otimes \lambda^{-1}, (\psi_E)_\Delta) &= \omega_{\Pi \otimes \lambda^{-1}}(\Delta)\,\gamma(\Pi \otimes \lambda^{-1}, \psi_E) \\
&= (\omega_\Pi \lambda^{-2})(\Delta)\gamma(\Pi \otimes \lambda^{-1}, \psi_E) \\
&= \mu(N_{E/F}(\Delta))\lambda(-1)\lambda^{-1}(N_{E/F}(\Delta))\gamma(\Pi \otimes \lambda^{-1}, \psi_F) \\
&= \gamma(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1)
\end{aligned}$$

and the proposition follows.

**Proposition 3.2.3** Let $\mu$ be a character of $F$ and let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$. Suppose $\Pi$ is $\mu$-distinguished. Then $\Pi$ is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{E/F}$.

That $\Pi$ is a base change lift follows from theorem 3.1.2. Let $\Pi$ be $\mu$-distinguished. Then $\Pi \otimes \mu$ is distinguished where $\mu$ is an extension of to $E$ and hence $(\Pi \otimes \mu) \sim (\Pi \otimes \mu)^\sigma$ i.e. $\Pi \simeq \Pi^\sigma \otimes (\mu\mu^\sigma)$. But $\Pi \sim \Pi \otimes \omega_\Pi$ and $\omega_\Pi = \mu \circ N_{E/F} = \mu\mu$. Thus it follows that $\Pi \cong \Pi^\sigma$.

What remains to be proved is the assertion on the central character. To this end we will make use of theorems 3.1.5 and 3.1.8.

In order to prove our assertion on the central character for the principal series and special representations of $GL_2(E)$ we need to show the following.

(i) The principal series representation $\Pi(\chi, \chi^\sigma)$ $(\chi \neq \chi^\sigma)$ of $GL_2(E)$ is $\chi|$ distinguished and not $\chi|_F\,\omega_{E/F}$ distinguished. (This is because $\Pi(\chi, \chi^\sigma)$ is the base change lift of a supercuspidal representation with central character $\chi|_F\,\omega_E$.)

(ii) The principal series representation $\Pi(\chi_1, \chi)$ of $GL_2(E)$ with $\chi_1 = \mu \circ N_{E/F}$, $\chi_2$ $_2 \circ N_{E/F}$ (where $\mu$, $\mu_2$ are characters of $F$) is $\mu_2$ distinguished as well as $\mu\,\mu_2\omega$ distinguished.

(iii) The representation $\Sigma(\chi|\,|_E^{\,2}, \chi|\,|_E^{\,1/2})$ with $\chi = \mu \circ N_{E/F}$ (where $\mu$ is a character of $F$) is $\mu^2\omega_{E/F}$ distinguished and not $\mu^2$ distinguished.

$\Pi(\chi, \chi^\sigma) \otimes \chi$ $\Pi(1, \chi \quad \chi^\sigma)$ is distinguished by theorem 3.1.5. Now take an extension $\widetilde{\omega_E}$ of $\omega$ to $E$ and consider $\Pi(\chi, \chi^\sigma) \otimes \chi \quad \widetilde{\omega}$. This is $\Pi(\widetilde{\omega_E} \quad \chi \quad \chi\,\omega)$ and since the restriction to $F$ of these two characters is not trivial, it follows by theorem 3.1.5 that $\Pi(\chi, \chi^\sigma) \otimes \chi \quad \widetilde{\omega_{E/F}}$ is not distinguished.

or equivalently $\Pi(\chi, \chi^{\sigma})$ is not $\chi|_F\, \omega_{E/F}$ distinguished

If $\Pi = \Pi(\chi_1, \chi_2)$ with $\chi_1 = \mu_1 \circ N_{E/F}$, $\chi_2 = \mu_2 \circ N_{E/F}$ ($\mu_1, \mu_2$ are characters of $F^*$) then for characters $\lambda$ of $E^*$ with $\lambda|_{F^*} = \mu_1\mu_2$

$$\begin{aligned}
\gamma(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1) &= \gamma(\chi_1\lambda^{-1}, \psi_E)\gamma(\chi_2\lambda^{-1}, \psi_E)\lambda(-1) \\
&= \gamma(\chi_1\lambda^{-1}, \psi_E)\gamma(\chi_1^{-1}\lambda^{\sigma}, \psi_E)\lambda(-1) \\
&= \chi_1\lambda^{-1}(-1)\lambda(-1) \\
&= \chi_1(-1) \\
&= 1
\end{aligned}$$

since $\chi_2\lambda^{-1} = \chi_1^{-1}\lambda^{\sigma}$ and $\chi_1 = \chi_1^{\sigma}$. The same argument works if we take $\lambda$ such that $\lambda|_{F^*} = \mu_1\mu_2\omega_{E/F}$. Thus $\Pi$ is both $\mu_1\mu_2$ distinguished and $\mu_1\mu_2\omega_{E/F}$ distinguished by proposition 3.2.2.

For a character $\mu$ of $E^*$ the special representation $\Sigma(\chi|\,|_E^{1/2}, \chi|\,|_E^{-1/2}) \otimes \mu$ is distinguished precisely when $\chi|_{F^*} = \omega_{E/F}$ by theorem 3.1.5. Hence $\Sigma(\chi|\,|_E^{1/2}, \chi|\,|_E^{-1/2}) \otimes \mu$ is distinguished precisely when $\mu|_{F^*} \cdot \chi|_{F^*}\omega_{E/F} = \mu^2\omega_{E/F}$. Thus $\Sigma(\chi|\,|^{1/2}, \chi|\,|^{-1/2})$ is $\mu^2\omega_{E/F}$ distinguished and not $\mu^2$ distinguished.

Now suppose that $\Pi$ is a supercuspidal representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$ which is $\mu$ distinguished. We must show that $\Pi$ is a base change lift of a representation of $GL_2(F)$ with central character $\mu\omega_{E/F}$. By what has already been shown $\Pi$ is a base change lift of a representation of $GL_2(F)$ (say $\pi$). Since $\omega_\Pi = \mu \circ N_{E/F}$, $\omega_\pi$ can be either $\mu$ or $\mu\omega_{E/F}$. What we need to show is that $\omega_\pi = \mu\omega_{E/F}$ and not $\mu$. This will follow from Saito's proof of Tunnell's formula [32]. Using the relation $\omega_\Pi = \mu \circ N_{E/F}$ in Saito's proof we finally get the identity

$$\chi_\pi(a) = \sum_{\lambda|_{F^*} = \mu} \frac{1 + \epsilon(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1)}{2}\lambda(a) + \sum_{\lambda|_{F^*} = \mu\omega_{E/F}} \frac{1 + \epsilon(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1)}{2}\lambda(a)$$

where $\chi_\pi$ is the character of $\tau$ and $a \in E^* - F^*$. Since $\gamma$ factor and $\epsilon$ factor are the same for supercuspidals and since $\Pi$ is given to be $\mu$ distinguished, $\epsilon(\Pi \otimes \lambda^{-1}, \psi_E)\lambda(-1) = 1$ for all characters $\lambda$ of $E^*$ with $\lambda|_{F^*} = \mu$ by proposition 3.2.2. Thus the first sum in the above identity vanishes and we get

$$\chi|_{(E^* - F^*)} = \sum_{\lambda|_{F^*} = \mu\omega_{E/F}} \frac{1+\epsilon(\Pi\otimes\lambda^{-1}, \psi_E)\lambda(-1)}{2}\lambda$$

Comparing with Tunnell's formula (Theorem 3.1.8) we have $\omega_\pi = \mu\omega_{E}$. This finishes the proof of proposition 3.2.3. Theorem 1.3.1 follows from the above propositions.

## 3.3 Proofs of theorems 1.3.2, 1.3.3 and 1.3.4

We now prove that statements (2) and (3) in theorem 1.3.2 are equivalent. Recall that $U(2,E/F) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(E) \mid w\,{}^tg^{\sigma} w = g\}$ where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $g^\sigma = \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix}$. Thus

$$U(2,E/F) = \{g \in GL_2(E) \mid \frac{1}{\sigma(det\ g)} g^\sigma = g\}$$

and the centre of $U(2,E/F)$ is $\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in GL_2(E) \mid N_{E/F}(a) = 1\}$. Therefore if a representation $\Pi$ of $GL_2(E)$ is distinguished with respect to $U(2,E/F)$, $\omega_\Pi$ factors through the norm map $N_{E/F}$. Define $GL_2^+(F)$ to be the subgroup of $GL_2(F)$ consisting of matrices whose determinant lies in $N_{E/F}E^*$. We observe that

$$Z(GL_2(F))GL_2^+(F) = Z(GL_2(F))\,U(2,E/F)$$

where $Z(GL_2(E))$ is the centre of $GL_2(E)$. Hence if $(\Pi, V)$ is $\mu$-distinguished and $l$ is a non zero linear functional on $V$ such that $l(\Pi(g)v) = \mu(\det g)l(v)$ for $g \in GL_2(F)$ then $l(\Pi(g)v) = l(v)$ for $g \in U(2, E/F)$. The case when $\Pi$ is $\mu\omega_{E/F}$-distinguished is similar. Conversely if $\Pi$ is distinguished for $U(2, E/F)$ and $l$ is a non zero linear functional on the space of $\Pi$ such that $l(\Pi(g)v) = l(v)$ for $g \in U(2, E/F)$, then $l(\Pi(g)v) = \mu(\det g)l(v)$ for $g \in GL_2(F)$. We define a new functional $l'$ on the space of $\Pi$ by

$$l'(v) = l(v) + \mu\omega_{E/F}(a)l\left(\Pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)v\right)$$

where $a \in F^* - N_{E/F}E^*$. Then it is easy to check that

$$l'(\Pi(g)v) = \mu(\det g)l'(v)$$

for $g \in GL_2(F)$. Thus if $l' \neq 0$ then $\Pi$ is $\mu$-distinguished. If $l' = 0$ then

$$l\left(\Pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)v\right) = \mu\omega_{E/F}(a)l(v)$$

and so $\Pi$ is $\mu\omega_{E/F}$-distinguished. The above together with the proof of theorem 1.3.1 completes the proof of theorem 1.3.2.

We now prove theorem 1.3.3. Let $\Pi$ be an irreducible admissible representation of $GL_2(E)$ with $\omega_{\Pi|F^*} = 1$. Since $\omega_{\Pi|F^*} = 1$ we have $\omega_\Pi = \eta/\bar{\eta}$ for a character $\eta$ of $E^*$ (by Hilbert 90). Note that

$$\omega \qquad \omega_\Pi$$

$$\eta_F \qquad \omega_{E/F}$$

Hence

$$\begin{aligned}
\Pi \text{ is dist or } \omega_{E\ F} \text{ dist} &\Leftrightarrow \Pi\otimes\eta \text{ is } (\eta|_F) \text{ dist or } (\eta|_F)\omega_{E} \text{ dist} \\
&\Leftrightarrow \Pi\otimes\eta \quad \text{a base change lift for } GL_2(F) \\
&\Leftrightarrow (\Pi\otimes\eta) \ \sim \Pi\otimes\eta \\
&\Leftrightarrow \Pi \ \sim \Pi\otimes \quad_{\eta} \\
&\Leftrightarrow \Pi \ \sim \Pi
\end{aligned}$$

This proves theorem 1 3 3

If a supercuspidal $\Pi$ is a base change of $\pi$ and $\tau$ then $\pi = \tau\otimes\omega_E$ In particular $\omega_\pi = \omega_\pi$ Therefore from the above argument we conclude that $\Pi$ is either distinguished (when $\Pi\otimes\eta$ is a base change of a representation of central character $\eta|_F\,\omega_{E\ F}$) or $\omega$ distinguished (when $\Pi\otimes\eta$ is a base change of a representation of central character $\eta|_F$) but not both This discussion together with theorem 3 1 5 proves theorem 1 3 4

# 4 FLICKER'S THEOREM: A LOCAL PROOF

## 4.1 *Prompting the theorem*

Let $E/F$ be a separable quadratic extension of global fields. In [13], Flicker deduces theorem 1.3.5 from a similar global theorem which roughly says that the cuspidal $GL_2(\mathbf{A}_F)$ distinguished representations of $GL_2(\mathbf{A}_E)$ are those in the image of the unstable base change map from the set of cuspidal non degenerate representations of $U(2, E/F)(\mathbf{A})$ whose image under the base change map is cuspidal.

The heuristics behind this theorem is the following. By the main theorem of [12], a cuspidal representation $\Pi$ of $GL_n(\mathbf{A}_E)$ is distinguished by $GL_n(\mathbf{A}_F)$ precisely if the twisted tensor $L$-function of $\Pi$ has a pole at $s = 1$. This, together with the Langlands functoriality principle, would imply that distinguished $\Pi$ correspond to those representations of the Weil group of $F$ (to the $L$ group of $GL_n(E)$) which, when composed with the twisted tensor representation of the $L$-group of $GL_n(F)$, contain the trivial representation (see p. 141, [13]). There it is also proved that when $n$ is even (resp. odd), the unstable (resp. stable) base change map from the $L$ group of $U(n)$ to the $L$ group of $GL_n(E)$ composed with the twisted tensor representation contains the trivial representation. So one would expect that $GL_n(F)$ distinguished cuspidal automorphic representations of $GL_n(E)$ are precisely the image of the unstable (resp.

stable) base change map from $U(n)$ for $n$ even (resp. odd).

Let us switch back to local notations. Observe that theorem 1.3.3 can be viewed as a dual to the equivalence of (1) and (2) in theorem 1.3.2. This duality can easily be seen in the simplest example of $E^\times = GL_1(E)$, $F^\times = GL_1(F)$, $U(1, E/F) = \{g \in E^\times \mid N_{E/F}(g) = 1\} = \{g/g^\sigma \mid g \in E^\times\}$. If a character $\chi$ of $E^\times$ is distinguished with respect to $F^\times$, then $\chi|_{F^\times} = 1$. By Hilbert 90, there exists a character $\eta$ of $U(1, E/F)$ such that

$$\chi(g) = \eta(g/g^\sigma)$$

or $\chi$ is a lift of a character of $U(1, E/F)$. Dually, if a character $\chi$ of $E^\times$ is trivial on $U(1, E/F)$, then we have $\chi = \chi^\sigma$, which implies

$$\chi = \eta \circ N_{E/F}$$

where $\eta$ is a character of $F^\times$. In other words, $\chi$ is a lift of a character of $F^\times$.

This duality can be observed in the finite field case too. Let $F$ be the finite field of $q$ elements and $E$ the field of $q^2$ elements. Let $\sigma$ denote the Frobenius morphism. Then it is known that an irreducible representation $\Pi$ of $GL_n(E)$ has a $GL_n(F)$ fixed vector if and only if $\Pi \sim \Pi^\sigma$ and $\Pi$ has a $U(n)$ fixed vector if and only if $\Pi \cong \Pi^\sigma$ [19].

## 4.2 Motivating the local proof

If $\Pi$ is an irreducible admissible representation of $GL_2(F)$ with $\omega_\Pi|_{F^\times} = 1$, then theorem 1.3.3 says that $\Pi \simeq \Pi^\sigma$ if and only if $\Pi$ is distinguished or $\omega_{E/F}$-distinguished. Flicker's theorem is a refinement of this theorem since it specifies when exactly such a $\Pi$ is distinguished.

To motivate our proof let us first consider the dual situation Recall the equivalence of (1) and (3) in theorem 1 3 2 Thus if $\mu$ is a character of $F$ and $\Pi$ an irreducible admissible representation of $GL_2(E)$ with $\omega_\Pi = \mu \circ N_{E/F}$ then $\Pi$ is a base change lift from $GL_2(F)$ if and only if $\Pi$ is distinguished or $\omega$ distinguished But in chapter 3 of this thesis we have refined this statement and we could specify precisely which $\Pi$ are $\mu$ distinguished These are the ones that base change from a representation of $GL_2(F)$ with central character $\mu\omega_E$ (see Theorem 1 3 1) A key ingredient involved in deriving this refinement was theorem 3 1 9 rather Saito's proof of it

This tells us that we might be able to use Saito's method to deduce Flicker's theorem as well That is what we do in the next section Before that let us summarize the main ideas involved here

Note that for principal series representations and special representations of $GL(E)$ theorem 1 3 5 will follow from the results due to Flicker and Hakim (Proposition 2 3 5 and Theorem 3 1 5) For supercuspidals we adopt the method due to Saito mentioned above to get the desired result Corresponding to the quadratic extension $E$ of $F$ we fix an embedding $\imath$ of $E/F$ in $U(2, E/F)$ given by

$$(aF) - \begin{pmatrix} x & y \\ \Delta^2 y & x \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{where } a = x + \Delta y \in E$$

If $g = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ then observe that $gg^{\tau}$ (where $\tau$ is the involution $g \to w\, g^{\sigma} w$) and $\imath(aF)$ are conjugate in $GL(E)$ By means of the base change theory of $U(2, E/F)$ we get two formulae one in the stable base change case and the other in the unstable base change case for $\chi_{\{\pi\}}(\imath(aF))$ where $\{\pi\}$ is the packet of representations of $U(2, E/F)$ that base changes to $\Pi$ and $\chi_\pi$ is the sum of characters of the representations in the packet of $\pi$. These formulae must be seen as the analogues of Tunnell's formula for characters of $GL(2)$. Flicker's theorem can be derived as a corollary to the proof of these formulae just as Saito deduces theorem 3.1.9 from his proof of Tunnell's formula (Theorem 3.1.8).

## 4.3 Proof of Flicker's theorem

As mentioned in the previous section, theorem 1.3.5 is verified for principal series and special representations once we appeal to proposition 2.3.5 and theorem 3.1.5. So let $\Pi$ be a supercuspidal representation of $GL_2(E)$.

Suppose $\Pi$ is a base change lift of a representation of $U(2, E/F)$. Then $\Pi \simeq \Pi$ and $\omega_{_\Pi}|_F = 1$, i.e. $\Pi \simeq \omega_{_\Pi} \otimes \Pi^\sigma$ and $\omega_{_\Pi}|_F = 1$. So by the uniqueness of the Kirillov model

$$\mathcal{K}(\Pi, \psi_E) = \mathcal{K}(\omega_{_\Pi} \otimes \Pi^\sigma, \psi_E).$$

Note that $I_\sigma$ defined on the Kirillov model $\mathcal{K}(\Pi, \psi_E)$ of $\Pi$ by

$$I_\sigma f(x) = \omega_{_\Pi}(x) f(x^\sigma)$$

gives an intertwining operator from $(\Pi, \mathcal{K}(\Pi, \psi_E))$ to $(\omega_{_\Pi} \otimes \Pi^\sigma, \mathcal{K}(\Pi, \psi_E))$. Also $I^2 =$ identity since $\omega_{_\Pi}|_F = 1$ and $\Pi(\sigma h) = I_\sigma \Pi(h) I_\sigma$. We extend $\Pi$ to $GL_2(E) \times Gal(E/F)$ by

$$\Pi(g, \sigma) = \Pi(g) I$$

Now we compute the value of the twisted character $\chi$ at

$$g = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$$

where $a \in L$ $\Delta \in E$ such that $Tr_{E\ F}(\Delta)$ 0

Since $\Pi$ is a supercuspidal representation $K(\Pi\ \omega_E)$ des with the space of Schwartz Bruhat functions $\mathcal{S}(F\ )$ on $E$ and a basis of this space is given by the set of following functions

$$\xi_\lambda\ (x) = \begin{cases} \lambda(x) & \text{if } v_F(x) \\ 0 & \text{othe s} \end{cases}$$

Here $n$ varies over all integers and $\lambda$ varies over a complete set of representatives of all characters of $E$ modulo $\sim$ where $\lambda \sim \lambda_2$ if and only if $\lambda\ \lambda_2$ is unramified We have the lemma (Lemma 2 1 [32])

Lemma 4 3 1 $\Pi(w)\xi_\lambda$ $\epsilon(\Pi \otimes \lambda\ \psi_E)\xi^{m}_{\omega_\Pi \lambda}$ where $n$ $f(\Pi \otimes \lambda\ ) + 2\ (\psi_F)$

Here $f(\Pi \otimes \lambda\ )$ and $(\omega_E)$ denote the conductor exponents of $\Pi \otimes \lambda$ and respectively Using this lemma we compute $\Pi(g)I$

$$\begin{aligned} \Pi(g) I_\sigma \xi_\lambda^{\ n} &= \Pi \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \Pi \begin{pmatrix} a & 0 \\ 0 & \end{pmatrix} \Pi(w) I_\sigma \xi \\ & \Pi \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \Pi \begin{pmatrix} a^{\ 1} & 0 \\ 0 & \end{pmatrix} \Pi(w) \xi_{\omega_\Pi \lambda} \\ & (\Pi \otimes \omega_{_\Pi}\ \ \lambda^{\ \sigma}\ \psi_F) \Pi \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \Pi \begin{pmatrix} a & 0 \\ 0 & \end{pmatrix} \xi \\ & (\text{where} \ \ - f(\Pi \otimes \qquad )\ \ 2\ (\quad)\ \ n) \\ & (\Delta)\ (\Pi \otimes \omega_{_\Pi}\ \ \lambda\ \ \psi_E)\lambda\ \ (\ a\ \ \ \xi_\lambda \end{aligned}$$

But

$$\Pi \otimes \omega_{\Pi} \quad \lambda^{\ \sigma} \sim \Pi \otimes \lambda^{\ \sigma} \sim \Pi^{\sigma} \otimes \lambda^{\ \sigma} = (\Pi \otimes \lambda \quad)^{\sigma}$$

Therefore

$$\epsilon(\Pi \otimes \omega_{\Pi} \quad \lambda^{\ \sigma} \ \psi_F) - \epsilon(\Pi \otimes \lambda \quad \psi_E) \text{ a } \mathrm{d} f(\Pi \otimes \omega \quad \lambda^{\ \sigma}) - f(\Pi \otimes \lambda \quad )$$

Thus

$$\Pi(g) I_{\sigma} \xi_{\lambda} \qquad \omega_{\Pi}(\Delta) \epsilon(\Pi \otimes \lambda \qquad _L) \lambda^{\sigma}( \quad ) \xi \qquad ^{E \ a}$$

where $m = f(\Pi \otimes \lambda^{\ 1}) + 2n(\psi_F) - n$ We have thus proved

Lemma 4 3 2 For $a \in E \quad F$

$$\Pi\left(\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w\right) I \ \xi_{\lambda} \qquad \omega_{\Pi}(\Delta) c(\Pi \otimes \lambda \quad \psi_E) \lambda^{\sigma}(-a) \xi_{\lambda}^{m \quad E \ a}$$

where $m \quad f(\Pi \otimes \lambda \quad ) + 2n(\psi_E) \quad n$

We want to compute $\chi_{\Pi} \ (g)$ where

$$g \quad \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} u$$

There is a standard method to do this and we refer to (p 102 103 [32]) for the details Set $\Gamma \ = \begin{pmatrix} 1 + \mathcal{P}_E & \mathcal{P}_F \\ \mathcal{P}_E & 1 + \mathcal{P}_E^n \end{pmatrix} \cap GL_2(\mathcal{O}_E)$ Let $\mathcal{K}(\Pi \ \psi_E)^n$ be the subspace of $\mathcal{K}(\Pi \ \psi_E)$ consisting of elements invariant under $\Gamma_n$ Let

$$B \ = \left\{ \xi_{\lambda}^{n} \quad \begin{matrix} \text{duc} \quad \text{f } \lambda < \\ f(\Pi \otimes \lambda \quad ) + n(\psi_F) \quad < \quad n( \ _E) + r \end{matrix} \right\}$$

Then $B_n$ gives a basis of $\mathcal{K}(\Pi\ \psi_E)^n$ for $n$ sufficiently large and $\bigcup_n B_n$ gives a basis of $\mathcal{K}(\Pi\ \psi_E)$ Let $P_n$ be the projection of $\mathcal{K}(\Pi\ \psi_E)$ onto $\mathcal{K}(\Pi\ \psi_E)^n$ defined by

$$\frac{\int \Pi(g)dg}{\int dg}$$

where $dg$ is a Haar measure on $GL_2(E)$ Then the value of $\chi\quad(g)$ can be calculated as $trace(\Pi(g)I_\sigma P_n)$ with respect to this basis for a sufficiently large n

Suppose $\xi_\lambda^{(n)}$ contribute to $\chi_\Pi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}w\right)$ Then we have

(i) $n\quad n(\psi_F)+\frac{1}{2}(f(\Pi\otimes\lambda^{\ 1})\quad v_E(a))$

(ii) $\lambda|_{\mathcal{O}_E} - \lambda^{\ \sigma}|_{\mathcal{O}_E}$

First assume that $E/F$ is unramified Then from (ii) we have $\lambda|_{\mathcal{O}_F}\quad 1$ As a representative of the class of $\lambda$ take $\lambda$ such that $\lambda(\varpi_F)-1$ Then we have $\lambda^{\ \sigma}-\lambda$ Thus the contribution to

$$\chi_\Pi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a^{\ 1} & 0\\ 0 & 1\end{pmatrix}w\right)$$

of $\xi_\lambda$ for the above $\lambda$ is equal to

$$\begin{cases}\omega_\Pi(\Delta)\epsilon(\Pi\otimes\lambda\quad\psi_E)\lambda^\sigma(\ a) & \text{if } \ _E(a)=f(\Pi\otimes\lambda\quad)(\text{mod}2)\\ 0 & \text{otherwise}\end{cases}$$

Since $E/F$ is unramified we have an extension $\widetilde{\omega_{F}}_{\ F}$ of $\omega_{E\ F}$ to $E$ which is unramified Then

$$\epsilon(\Pi\otimes\lambda\quad\omega_{E\ F}\quad\psi_E)\quad(\ 1)^{f(\Pi\otimes\lambda\quad)}\epsilon(\Pi\otimes\lambda^{\ 1}\ \psi_E)$$

Therefore the contribution of $\xi_\lambda^{(n)}$ to $\lambda_\Pi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}w\right)$ is equal to

$$\frac{1}{2}\omega_\Pi(\Delta)(\epsilon(\Pi\otimes\lambda\quad\ _F)\lambda^\sigma(\ a)+\epsilon(\Pi\otimes\lambda\ \widetilde{\omega}\quad\ _E)(\lambda\widetilde{\omega_E}\ )^\sigma(-a))$$

Thus

$$\chi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix} & 0\\ 0 & 1\end{pmatrix}v\right)$$

$$\omega(\Delta)\left(\sum \frac{\epsilon(\Pi\otimes\lambda \quad \psi_E)}{2}\lambda^{\sigma}(\quad) + \sum_{\lambda \quad \omega} \frac{(\Pi\otimes\lambda \quad \psi \quad)}{2}\lambda \quad a)\right)$$

$$\sum \frac{\epsilon(\Pi\otimes\lambda \quad (\psi_L)_\Delta)}{2}\lambda^{\sigma}(a) + \sum_{\lambda \quad \omega} \frac{\epsilon(\Pi\otimes\lambda \quad (\psi \quad )_\Delta}{2}\lambda \quad a)$$

$$\sum \frac{1+\epsilon(\Pi\otimes\lambda \quad (\psi_L)_\Delta)}{2}\lambda^{\sigma}(a) + \sum_{\lambda \quad \omega_F} \frac{1+\epsilon(\Pi\otimes\lambda \quad (\quad_F)_\Delta)}{2}\lambda \quad a)$$

since $\sum_{\lambda|} \lambda = 0$ and $\sum_{\lambda \quad \omega_{E/F}} \lambda^{\sigma} \quad 0$

Now suppose $E/F$ is a ramified extension. Let $\varpi$ be a uniformizing element of $F$ that is contained in the norm of $E$. The condition (ii) implies $\lambda|_{N_E \quad \mathcal{O}} \quad 1$. Therefore $\lambda|_{\mathcal{O}_F} \quad 1$ or $\omega_{E/F}|_{\mathcal{O}}$. In the class of $\lambda$ satisfying (i) there are exactly two characters satisfying $\lambda(\varpi_F) = 1$ $(i = 1, 2)$ and they satisfy $\lambda = \lambda^{\sigma}$. Since $\lambda$ and $\lambda_2$ belong to the same class $\lambda\lambda_2$ is unramified. Hence $\lambda\lambda_2(\varpi_F) \quad \lambda\lambda(\varpi_F) \quad 1$. This is $\lambda\lambda_2(\varpi_F) \quad \pm 1$. But $\lambda\lambda(\varpi \quad) \quad 1$ implies that $\lambda = \lambda_2$ which is not true and so $\lambda(\varpi_E) = \lambda_2(\varpi_E)$. Thus $\lambda_2 = \lambda\eta$ where $\eta(x) \quad (-1)^{E}$. Now the contribution of $\xi^n$ to $\chi_\Pi\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}u\right)$ is

$$\frac{1}{2}\omega_\Pi(\Delta)(\epsilon(\Pi\otimes\lambda \quad \psi_L)\lambda^{\sigma}(\quad a) + \epsilon(\Pi\otimes\lambda_2 \quad \psi_E)\lambda_2^{\sigma}(\quad a))$$

and $\lambda| \quad \lambda_2|$. But $\lambda|$ can be 1 or $\omega$. So the total contribution is

ng

$$\sum \frac{\epsilon(\Pi\otimes\lambda \quad (\psi_L)_\Delta)}{2}\lambda(a) + \sum_{L \quad F} \frac{(\Pi\otimes\lambda \quad (\psi_E)_\Delta)}{2}\lambda^{\sigma}(a)$$

Thus as in the ramified case we get

$$\chi\left(\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}u\right)\right)$$

$$\sum \frac{1+\epsilon(\Pi\otimes\lambda\quad(\psi_E)_\Delta)}{2}\lambda\ (a)+\sum\frac{1+\ (\Pi\otimes\lambda\quad(\psi\ )_\Delta)}{2}\lambda\ (a)$$

Fix an embedding of $E\ /F$ into $U(2\ E/F)$ given by

$$\imath(aF\ )\quad\begin{pmatrix}x & y\\ \Delta^2 y & x\end{pmatrix}\begin{pmatrix} & 0\\ 0 & a\end{pmatrix}\quad\text{where } a\quad x+\Delta y\in E$$

If $g = \begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}w$ then observe that $gg = \begin{pmatrix}a\ a^\sigma & 0\\ & 0 & 1\end{pmatrix}$ is conjugate to the image of $F$ under

If $\Pi$ is a stable base change of a representation $\tau$ of $U(2\ E/F)$ then

$$\chi_\Pi\left(\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\right)\quad\lambda\ (\ aF\ ))$$

and if $\Pi$ is an unstable base change of $\tau$ then

$$\chi_\Pi\left(\left(\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}w\right)\right) - \widetilde{\omega_{E\ F}}(a\quad)\chi\ (\ (aF\ ))$$

Thus we get the following two identities, the first obtained when $\Pi$ is a stable base change and the second when $\Pi$ is an unstable base change

$$\chi\ (\ (aF\ ))\qquad\sum\frac{+\ (\Pi\otimes\lambda\qquad)_\Delta}{2}\lambda^\sigma\ )$$

$$+\quad\sum\frac{1\quad\Pi\otimes\lambda\quad(\ _F)_\Delta)}{\ }\lambda\ (a)$$

$$\chi_{\{\ \}}(\imath(aF^*)) = \widetilde{\omega_{E/F}}(a)\sum_{\lambda}\frac{1+\epsilon(\Pi\otimes\lambda, (\psi_E)_\Delta)}{2}\lambda^{\sigma}(a) + \omega_{E/F}(\ )\sum\frac{1+\epsilon(\Pi\otimes\lambda, (\psi_E)_\Delta)}{2}\lambda(a)$$

Let $r \in F^* - N_{E/F}(E^*)$ and change $a$ to $ra$. In both these identities the left side remains unchanged whereas a change of sign occurs in the second sum of the first identity and in the first sum of the second identity. Thus it follows that the second sum vanishes in the first identity and the first sum vanishes in the second identity. Thus we get $\gamma(\Pi\otimes\lambda, (\psi_E)_\Delta) = 1$ for all characters $\lambda$ of $E^*$ with $\lambda|_{F^*} = \omega_{E/F}$ (respectively $\lambda|_{F^*} = 1$) if $\Pi$ is a stable (resp. unstable) base change lift of a representation of $U(2, E/F)$. (Note that the $\gamma$ factor is the same as the $\epsilon$ factor since $\Pi$ is supercuspidal.) Hence if $\Pi$ is a stable base change, then it is $\omega_{E/F}$ distinguished and if $\Pi$ is an unstable base change then it is distinguished (by Theorem 3.1.4).

What remains to show in order to prove theorem 1.3.5 is that a representation $\Pi$ of $GL_2(E)$ distinguished with respect to $GL_2(F)$ is obtained by the unstable base change map. Now by theorem 1.3.3 we know that $\Pi$ is a base change lift of a representation of $U(2, E/F)$. We must show that $\Pi$ is in the image of the unstable base change map and not in the image of the stable base change map. Suppose $\Pi$ is a stable base change lift of a representation of $U(2, E/F)$. Then by what has been proved already $\Pi$ is $\omega_{E/F}$ distinguished with respect to $GL_2(F)$. Thus $\Pi$ is both distinguished and $\omega_{E/F}$ distinguished which contradicts theorem 1.3.4. This proves theorem 1.3.5.

# 5 DISTINGUISHEDNESS FOR $GL(N)$

## 5.1 A multiplicity one result and applications

We know from theorem 3.1.1 that the space of $GL\ (F)$ invariant linear forms on the space of an irreducible admissible representation Π of $GL_n(E)$ has dimension at most one. In this section we look at a larger space which is the space of $P_n\ F)$ inva nt linear forms on Π. We prove that when Π is a distinguished supercuspidal or when it is parabolically induced from two distinguished supercuspidals then the latter space has dimension one (Theorem 1.3.6). As mentioned in the introduction the proof follows essentially from the main lemma of [15].

Let $\tau$ Π$_{P\ E}$. Then has the Bernstein Zelevinsky filtration by $P\ (\Gamma)$ modules (see section 2.1)

$$0 \subset \quad \subset \tau_n \quad \subset \quad \subset \quad -$$

Thus

$$\tau \sim \oplus^{k\ n}\ {}_{k}/\ {}_{k} \quad \sim \oplus_{k}^{k}{}_{1}^{n}(\Phi\ )^{k}\quad \Psi\ (\tau^{\ k}\ )$$

If Π is supercuspidal by proposition 7 we conclude that

$$\sim (I\ )\quad \Psi\ (1$$

If $\Pi$ $I(\Pi\ \ \Pi_2)$ with $\Pi$ distinguished supercuspidals (incidentally such a $\Pi$ is distinguished by proposition 26 of [14]) use proposition 2 1 7 and proposition 2 1 8 to conclude that

$$\sim(\Phi^+)^n\ \ \Psi\ (\Pi\ )\oplus(\Phi\ )\ \ \Psi^+(\Pi\ )\oplus\ \Psi\ )\ \ \Psi\ \ (1)$$

Let now switch to Flicker's notation [15] Put $H\ (E)$ $GL\ (E)\Lambda_n(E)$ where $GL\ (F)$ embeds in $GL\ (E)$ via

$$g\mapsto\begin{pmatrix} g & 0\\ 0 & I\end{pmatrix}\quad 0\le j<n$$

Then $H\ (E)$ consists of

$$\begin{pmatrix} g & x\\ 0 & u\end{pmatrix}\quad g\in GL\ (E\ \ u\in N\ \ (E)$$

If $\rho$ is a representation of $GL_j(E)$ then $\rho\otimes u$ denotes the representation of $H_j(E)$ on the space of $\rho$ on which $\begin{pmatrix} g & x\\ 0 & u\end{pmatrix}$ acts by $\rho(g)\psi(u)$ Then

$$(\Phi\ )^n\ \ \Psi\ (\rho)=ind_{H\ \ E}^{P\ \ F}(\ \ ^{-1}\otimes\rho\otimes\ \ )$$

where ind is non normalized compact induction Now Flicker proves that the dimension of $Hom\ \ _F((\Phi^+)^n\ \ {}^1\Psi^+(\rho)\ 1)$ is equal to the dimension of $Hom\ \ _F(\ |_E^{-1}\otimes\rho\ 1)$ In particular

$$Hom\ \ ((\Phi\ )\ \ \Psi\ (\ )\ 1-\begin{cases}0 & \text{if } j\neq 0 \text{ and } \rho \text{ is distinguished}\\ \mathbb{C} & \text{if } j=0 \text{ and } \rho\ \ 1\end{cases}$$

This proves theorem 1 3 6

Now consider the linear form $l$ defined on the Whittaker model $\mathcal{W}(\Pi\ \psi)$ of $\Pi$ by

$$l(W) - \int_{N\ (\Gamma)\backslash P\ (F} W(p)d_r p$$

where $d\ p$ is a $P_n(F)$ right invariant measure on $N_n(\Gamma)\backslash P_n(F)$

The integral in the definition of the linear form displayed above converges (see lemma on p 306 in [12]) Moreover this linear form is non zero A vector on which $l$ is non zero can be chosen using the Iwahori factorization If $\Pi$ is a distinguished representation of $GL_n(E)$ as in theorem 1 3 6 we claim that the linear form $l$ defined above is the distinguished (i e non zero $GL_n(F)$ invariant) functional upto a scalar on the Whittaker model of $\Pi$

Proposition 5 1 1 Let $\Pi$ be as in theorem 1 3 6 Let $l$ be a distinguished functional on the Whittaker model of $\Pi$ Then $l$ can be normalized so that $l(W)$ $\int_{N_n(F\ \ P_n(\Gamma} W(p)d\ p$

Proof If $l$ is a distinguished functional on $\mathcal{W}(\Pi\ \psi)$ then in particular $l \in Hom_{P\ \ \Gamma)}(\Pi\ 1)$ On the other hand $\int_{N\ (F\ \ P\ (\ )} W(p)d_r p$ is also a $P_n(\Gamma)$ invariant functional on $\mathcal{W}(\Pi\ \psi)$ Thus both differ just by a scalar by theorem 1 3 6

Now we prove theorem 1 3 7 Call

$$l(W) \quad \int_{N\ \ \Gamma\ \ P_n\ \Gamma)} W(p)d\ p$$

and

$$l(W) \quad \int_{N\ \ \Gamma\ \ P\ \ \Gamma} W(p\ d\ p$$

Then $l$ is a non zero $P\ (F)$ invariant form on $\mathcal{W}(\Pi\ \psi)$ and $l$ is a non zero $P\ (F)$ invariant form on $\mathcal{W}(\Pi\ \ \psi\ \ )$ Suppose $l(W) = l(W)$ for all $W$

Then for any $p \in P_n(F)$

$$l(\Pi( p^{\ 1})W ) = l(\Pi(\widetilde{{}^tp\ \ })W ) \quad l(\Pi(p)W) \quad l(W) = l(W)$$

Thus $l$ is invariant under the transpose of $P_n(F)$ as well Since $\omega_\Pi|_F$ 1 we conclude that $l$ is invariant under the standard maximal parabolic and its transpose Since these together generate $GL_n(F)$ see Corollary 2 1 2) $l$ is a $GL\ (F)$ invariant functional and hence $\Pi$ is distinguished

Conversely suppose $\Pi$ is distinguished Then $\Pi$ is also distinguished By proposition 5 1 1 $l$ and $l$ are distinguished functionals on $\mathcal{W}(\Pi\ \psi)$ and $\mathcal{W}(\Pi\ \psi\ \ )$ respectively Therefore if we define $l$ on $\mathcal{W}(\Pi\ \psi)$ by

$$l\ (W) = l(\widetilde{W})$$

then both $l$ and $l$ define Haar integrals on $N_n(F)\backslash P_n(F)$ Thus $l(W\ )$ and $l(W)$ differ by a positive constant say $c$ Now since $W\ \ \ W$ we see that $c^2$ 1 This implies $c$ 1 since $c$ is positive Hence $l(W)$ $l(W\ )$ for all $W$

## 5 2 The poles of the local twisted tensor $L$ function

In this section we prove theorem 1 3 8 In fact we prove the following proposition and theorem 1 3 8 follows from it by proposition 5 1 1

Proposition 5 2 1 Let $\Pi$ be an irreducible admissible generic representation of $GL\ (E)$ which is distinguished with respect to $GL_n(F)$ Suppose $l(W) - \int_{N\ F\ P\ F} W\ (p)d\ $ is a $GL_n(F)$ invariant linear functional on $\Pi$ Then $s$ 0 is a pole of $L(s\ \ (\Pi\ )$

Proof. Call $G = GL_n(F)$, $H_2 = P_n(F)$, $H_1 = N_n(F)$. We apply lemma 2.4.1 to this triplet. So let $\nu_2 = d_r p$ be the $P_n(F)$ right invariant measure on $N_n(F)\backslash P_n(F)$. Let $\rho_2(g) = |\det g|_F$. Then $\rho_2$ is a rho function for $P_n(F)$ since $\Delta_{GL_n(F)} = 1$ and $\Delta_{P_n(F)}(p) = |\det p|_F$. As in lemma 2.4.1 let $\mu_1$ and $\mu_2$ be the associated quasi-invariant measures on $N_n(F)\backslash GL_n(F)$ and $P_n(F)\backslash GL_n(F)$. It can be verified that if $dg$ is the quotient measure on $N_n(F)\backslash GL_n(F)$ then $d\mu_1(g) = |\det g|_F dg$.

Let $\Pi$ be a distinguished supercuspidal representation of $GL_n(E)$ or $\Pi = I(\Pi_1, \Pi_2)$ with $\Pi_i$ distinguished supercuspidals of $GL_{n_i}(E)$, $n = n_1 + n_2$, $i = 1, 2$. We need to prove that $s = 0$ is a pole for $L(s, r(\Pi))$. Now consider

$$\Psi(1, W, \Phi) = \int_{N_n(F)\backslash GL_n(F)} \widetilde{W}(g)\Phi(\eta g)|\det g|_F \, dg$$

where

$$W \in \mathcal{W}(\Pi, \psi),\ \Phi \in \mathcal{S}(F^n) \text{ and } \eta = (0, 0, \ldots, 1).$$

By lemma 2.4.1 and theorem 1.3

$$\begin{aligned}
\Psi(1, W, \Phi) &= \int_{P_n(F)\backslash GL_n(F)} \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(pg)\Phi(\eta pg)\, d_r p \, d\mu_2(g) \\
&= \int_{P_n(F)\backslash GL_n(F)} \Phi(\eta g) \left(\int_{N_n(F)\backslash P_n(F)} \widetilde{W}(pg) d_r p\right) d\mu_2(g) \\
&= \int_{P_n(F)\backslash GL_n(F)} \Phi(\eta g) d\mu_2(g) \times \int_{N_n(F)\backslash P_n(F)} W(p) d_r p \\
&= c\, \Phi(0)\, l(W)
\end{aligned}$$

where $c$ is a non zero constant.

Next, as in (Proposition 4, [12]), we choose $W$ in $\mathcal{W}(\Pi, \psi)$ and $\Phi$ in $\mathcal{S}(F^n)$ such that $\Psi(s, W, \Phi)$ is identically one. This is done as follows (see p. 309, [12]).

Fix a congruence subgroup $K_m$ of $K$. By Iwahori factorization, we get a $\phi$ which is supported on $N_n(E)(K_m \cap P_n(E))$ and right invariant under $K_m \cap P_n(E) \cap B_n(E)$

where $B_n(E)$ is the group of upper triangular matrices. Fix $W$ in $\mathcal{W}(\Pi,\psi)$ with $W|_{P_n(F)} = \phi$. Then

$$\int_{N_n(F)\backslash P_n(F)} W(p)\,|\det p|_F\,dp = c\int_{N_n(F)\backslash P_n(F)} \phi(p)|\det p|_F^{1}\,dp$$

$c$ a non zero constant. Let $K_m$ denote the group of $k$ in $K$ with

$$\eta k = (\varpi^m x_1, \varpi^m x_2, \ldots, \varpi^m x_{n-1}, 1+\varpi^m x_n)$$

where $x_i$ are all in the ring $\mathcal{O}_E$. Choose $m$ so that $W$ is right invariant under $K_m \cap U_n(F)$ where $U_n(F)$ is the unipotent radical of the parabolic subgroup of type $(n-1,1)$. Let $\Phi$ be the characteristic function of the last row of $K_m \cap GL_n(F)$. Then $\Phi$ lies in $\mathcal{S}(F^n)$ and

$$\Psi(s,W,\Phi) = c\int dk \int_{N_n(F)\backslash P_n(F)} W(pk)\,|\det p|\,dp \int_{F} \Phi(\eta a k)\,|\det a|_F^{n}\,d^\times a$$

$c$ a non zero constant.

Now let us write down the functional equation at $s = 0$:

$$\Psi(1,W,\Phi)/L(1,r(\Pi)) = \epsilon(0,r(\Pi),\psi)\Psi(0,W,\Phi)/L(0,r(\Pi))$$

Choose $W$ and $\Phi$ as above. Then since $\Phi(0) = 0$, the left hand side of the functional equation vanishes. Since $W$ and $\Phi$ are such that $\Psi(0,W,\Phi) \neq 0$, it follows that $s = 0$ is a pole for $L(s,r(\Pi))$.

This proves proposition 5.2.1 and hence theorem 1.3.8.

## 5.3 Remarks on the local factors of distinguished representations

First let us prove theorem 1.3.9. This result follows from a result due to Bushnell and Henniart [8] which is stated below.

Theorem 5.3.1 Let $\pi$ be an irreducible admissible representation of $GL_n(F)$ with central character $\omega$. Let $\psi$ be a character of $F$. Then

$$\epsilon(\tfrac{1}{2}, \pi \times \pi, \psi) = \omega_\pi(-1)$$

Now let $\Pi$ be an irreducible tempered representation of $GL_n(E)$ and let $\pi$ be its lift under automorphic induction (see section 2.3). Then $\Pi^\sigma$ also lifts to $\pi$ and $\Pi$ lifts to $\pi$ (Proposition 2.3.3). Thus by proposition 2.3.4 we have

$$\epsilon(\frac{1}{2}, \pi \times \pi, \psi) = \epsilon(\frac{1}{2}, \omega_{E/F}, \psi)^{2n} \epsilon(\frac{1}{2}, \Pi \times \Pi^\sigma, \psi_E)\epsilon(\frac{1}{2}, \Pi \times \Pi, \psi_E)$$

By theorem 5.3.1 we get

$$\epsilon(\frac{1}{2}, \pi \times \pi, \psi) = \omega_\pi(-1)^{2n\ 1}$$

and

$$\epsilon(\frac{1}{2}, \Pi \times \Pi, \psi_E) = \omega_\Pi(-1)^n$$

Also

$$\epsilon(\frac{1}{2}, \omega_{E/F}, \psi)^2 = \omega_{E/F}(-1)$$

But by proposition 2.3.3 we know that

$$\omega_\pi = \omega_{E/F}\,\omega_\Pi|_F$$

Therefore

$$\begin{aligned}\epsilon(\tfrac{1}{2}, \Pi \times \Pi^\sigma, \psi_E) &= \frac{[illegible]}{\omega_{E/F}[illegible]^{2n}\,\omega_\Pi(-1)^n} \\ &= \frac{\omega_{E/F}[illegible]\,\omega_\Pi[illegible]}{\omega_{E/F}[illegible]^{2}[illegible]} \\ &= \omega_\Pi(-1)^n\end{aligned}$$

Since

$$\omega\ \omega_{\Pi}^{\sigma}(\Delta) = \omega(-1)$$

we get

$$\epsilon(\frac{1}{2}, \Pi \times \Pi^{\sigma}) = 1$$

by proposition 2.2.2 (ii). Now if Π is also distinguished then by theorem 3.1.2 $\Pi \sim \Pi^{\sigma}$ and therefore from what has been [illegible] already it follows that

$$\epsilon(\frac{1}{2}, \Pi \times \Pi) = 1$$

Now suppose $\Pi_1$ and $\Pi_2$ are two distinguished irreducible admissible representations of $GL_n(E)$ and $GL_n(E)$ respectively. By proposition 2.2.2 and theorem 3.1.2 we see that

$$\epsilon(\frac{1}{2}, \Pi_1 \times \Pi_2) = \pm 1$$

It is likely that when $\Pi_i$ are distinguished this epsilon factor is actually one. In some particular cases we have seen that this is true (Theorem 3.1.4, Corollary 3.[illegible].6, Theorem 1.3.9). If one believes in the conjectural proposition 2.2.6 he may believe it is worth analyzing the twisted epsilon factor $\epsilon(\frac{1}{2}, r(\Pi), \psi)$ for a distinguished Π. By proposition 2.2.5 we know that for a distinguished Π

$$\epsilon(\frac{1}{2}, r(\Pi), \psi) = \pm 1$$

The question is whether or not we can fix the sign in the above equation. Theorem 5.3.1 does precisely this when $E = F \oplus F$. The proof of theorem 5.3.1 (see [8]) seems to be telling us that a better understanding of the twisted tensor $L$ [illegible] is necessary in order to attempt this question.

A more speculative question (see Theorem 3.1.4) would be to ask for a converse theorem for distinguished representations of $GL_n(E)$. Let $\Pi$ be an irreducible admissible representation of $GL_n(E)$ with $\omega\ |_F$ 1 and suppose it is given that

$$\gamma(\frac{1}{2}, \Pi \otimes \Pi' \qquad 1$$

for all distinguished $\Pi$ of $GL\ \ E)$ where $1 \leq j < \ -1$. Can conclude that $\Pi$ is distinguished?

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